Number of Firms and Price Competition*

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February, 2014

Abstract

The relationship between the number of firms and price competition is a central issue in economics. To explore this relationship, we modify Varian’s (1980) model and assume that firms are privately informed about their costs of production. Allowing that the support of possible cost types may be large, we show that an increase in the number of firms induces lower (higher) prices for lower-cost (higher-cost) firms. We also characterize the pricing distribution as the number of firms approaches infinity, finding that the equilibrium pricing function converges to the monopoly pricing function for all but the lowest possible cost type. If demand is inelastic, an increase in the number of firms raises social welfare. If in addition the distribution of types is log concave, then an increase in the number of firms raises aggregate consumer surplus and lowers producer surplus. We identify conditions, however, under which uninformed consumers are harmed, and informed consumers are helped, when the number of firms is larger. By contrast, when the number of firms is held fixed, a policy that increases the share of informed consumers benefits informed and uninformed consumers. Finally, we confirm that results previously obtained in Varian’s (1980) complete-information model can be captured in our model as a limiting case when the support of possible cost types approaches zero.

Keywords and Phrases: Number of firms, price competition, welfare

JEL Classification Numbers: D43, D82, L13

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*We thank Sungbae An, Jackie Chan, Sjaak Hurkens, Weerachart Kilenthong, Art Tosborvorn, Julian Wright, Zhenlin Yang, Vasileios Zikos and seminar participants at NUS, NTU and SMU in Singapore and UTCC in Thailand. We also thank participants at the 2009 Far East and South Asia Meeting of the Econometric Society and the 2012 EARIE conference.

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1 Introduction

We consider a model of price competition among firms selling a homogeneous good. The model has two key features: firms are privately informed as to their respective costs, and consumers are heterogeneously informed about the prices in the market. In this context, we address several basic questions: Does price competition become more intense as the number of firms increases? Do lower- and higher-cost firms respond differentially to an increase in the number of firms? How does new entry affect consumer surplus, producer surplus and social welfare? Does the impact of entry on a consumer’s surplus depend upon the extent to which the consumer is informed about available prices?

Our model is a modified version of Varian’s (1980) model. Varian (1980) assumes that consumers are heterogeneous in the following sense: a fraction of consumers are informed about prices and purchase from the firms(s) with the lowest price, while the remaining fraction are uninformed about prices and randomly choose firms from which to purchase. Varian’s (1980) model is characterized by a unique symmetric mixed-strategy equilibrium. This equilibrium reflects the tradeoff that a firm faces between reducing its price so as to increase the firm’s probability of winning the informed consumers and increasing its price so as to earn more on the firm’s captive stock of uninformed consumers. Following Spulber (1995), Bagwell and Wolinsky (2002) and Bagwell and Lee (2010), we modify Varian’s model to allow that firms have heterogeneous cost types drawn from an interval and are privately informed about their respective cost types.\footnote{Our model is also distinct from Varian’s (1980), since we take the number of firms to be exogenous. Assuming a positive fixed cost of entry and using a free-entry condition, Varian (1980) endogenizes the number of firms.} We assume further that a cost type for a firm determines that firm’s constant marginal cost of production. A consequence of our incomplete-information approach is that the unique symmetric pricing equilibrium takes the form of a pure strategy.\footnote{Spulber (1995) establishes the existence of a unique equilibrium when all consumers are informed, while Bagwell and Wolinsky (2002) and Bagwell and Lee (2010) assume that some consumers are uninformed. Further discussion of this work is provided at the end of Section 2.}

An important benefit of our model is that we can examine how a firm’s response to an increase in the number of firms, \( N \), depends upon the firm’s cost type. An increase in \( N \) generates both price-decreasing and price-increasing forces. On the one hand, an increase in \( N \) may lead to lower prices, as some firms compete more aggressively for informed consumers. On the other hand, an increase in \( N \) may lead to higher prices, as some firms become discouraged about the prospect of winning informed consumers and thus focus on sales to uninformed consumers. Our analysis reveals that the balance of these competing forces hinges on a firm’s cost type: as \( N \) rises, lower-cost firms reduce prices while higher-...
cost firms increase prices. An interesting implication of this finding is that the potential extent of price dispersion, as measured by the support of possible equilibrium prices, is larger in markets with a greater number of firms.

We also characterize the equilibrium pricing function when the number of firms approaches infinity. As $N$ gets sufficiently large, the cutoff type above which firms increase prices is close to the lowest possible cost type, the prices for all types above the cutoff type are approximately equal to their respective monopoly prices, and the prices for types near the lowest possible cost type are approximately equal to their respective marginal costs. Thus, as $N$ approaches infinity, the equilibrium pricing function converges to the monopoly pricing function for all but the lowest possible cost type whose price converges to marginal cost.

The equilibrium pricing function also varies interestingly with the share of informed consumers, $I \in (0,1)$. While an increase in $N$ lowers the price of only lower-cost firms, an increase in $I$ reduces the price of all cost types other than the highest possible cost type. As $I$ increases, the price of a firm with the highest possible cost type remains fixed at the monopoly level. Thus, if the number of firms is held fixed, then a policy that increases the share of informed consumers benefits uninformed and informed consumers.

We next analyze the welfare implications of an increase in $N$. Our formal results here are obtained in the setting with inelastic demand. For any distribution function of cost types, we find that an increase in $N$ increases social welfare. Intuitively, an increase in $N$ lowers the expected cost of the firm that sells to informed consumers. Further, if the distribution function of cost types is log-concave, then an increase in $N$ raises aggregate consumer surplus and lowers aggregate producer surplus.³ We also consider the impact of an increase in $N$ on the respective surpluses of informed and uninformed consumers. Importantly, we identify conditions under which uninformed consumers are harmed when the number of firms is larger. First, we show that, for any $N$, there exists $N^* > N$ such that uninformed consumers enjoy higher consumer surplus with $N$ firms than for any number of firms in excess of $N^*$. Second, we establish that an increase in $N$ harms uninformed consumers if the support of possible cost types is sufficiently small. The key intuition is that a larger number of firms may discourage price competition by higher-cost firms and thereby increase the expected price paid by uninformed consumers. By contrast, informed consumers purchase from the lowest-cost firm in the market and gain under the same conditions when the number of firms is larger. We also provide numerical examples with related findings when $N$ is small and the support of possible cost types may be large.

To further relate our findings to those in the literature, we provide as well a purification

³In fact, as we show, an increase in $N$ raises aggregate consumer surplus under a weaker distributional assumption.
result: the pure-strategy equilibrium distribution of prices in our incomplete-information model corresponds to the mixed-strategy equilibrium distribution of prices in Varian’s (1980) complete-information model, when the support of possible costs in our model is sufficiently small. From this perspective, we may relate our findings to those of Morgan, Orzen and Sefton (2006). They study Varian’s (1980) model and show that an increase in $N$ lowers the surplus of uninformed consumers and increases the surplus of informed consumers. The counterpart results in our model obtain when the support of possible cost types is sufficiently small. As noted above, however, we also show that a sufficiently large increase in the number of firms harms uninformed consumers even when the support of possible cost types may be large so that the purification result does not apply. A further distinction is that our model provides a pure-strategy analysis in which an increase in $N$ affects differentially the pricing of lower- and higher-cost firms. Morgan, Orzen and Sefton (2006) also provide novel experimental evidence in support of the general hypothesis that an increase in $N$ harms uninformed consumers but benefits informed consumers.

The conventional view that more firms leads to more competitive pricing has been challenged in other models as well. Rosenthal (1980) works with a Varian-style (1980) model and shows that an increase in $N$ may harm consumers. Differently from our analysis, however, he assumes that an increase in the number of firms is paired with an increase in the share of uninformed consumers, so that the number of uninformed consumers per firm is invariant. Stahl (1989) modifies Varian’s (1980) model to allow for costly sequential search by uninformed consumers. He also finds that an increase in $N$ generates price-decreasing and price-increasing forces, where the latter force strengthens relative to the former as $N$ increases. Janssen and Moraga-Gonzalez (2004) explore related themes when uninformed consumers engage in costly non-sequential search. In equilibria in which uninformed consumers search for only one price, an increase in $N$ raises the expected price; however, this type of equilibrium fails to exist in their model if $N$ is large. Finally, Chen and Riordan (2007, 2008) and Perloff, Suslow and Seguin (2005) find that, when new entry entails product differentiation, more consumer choice associated with entry may lead to less competitive pricing. Importantly, all of these papers focus on the heterogeneity of consumers and assume that firms have the same production costs. In contrast, our paper allows that firms have heterogeneous costs and are privately informed about their respective cost types.\footnote{Reinganum (1979) considers a model of price dispersion when firms have heterogeneous costs and are privately informed about their costs levels; however, in her model, there are no informed consumers, since all consumers must search to learn price information.}

Our analysis of competing forces is also related to other analyses of incomplete-information games. Hopkins and Kornienko (2007) use a price equilibrium similar to ours...
and discuss two competing effects with reference to shifts in the cost distribution: if the cost distribution shifts to make lower-cost types more likely in the sense of the monotone likelihood ratio order, then lower-cost firms compete more intensely while higher-cost firms become discouraged and compete less intensely. Bagwell and Lee (2010) focus primarily on a model of advertising competition among firms that are privately informed as to their costs. They construct an advertising equilibrium in which lower-cost firms advertise at higher levels and find that an increase in the number of firms generates competing forces for the intensity of advertising competition.

This paper is organized as follows. Section 2 introduces the model and characterizes the pricing equilibrium. Section 3 provides comparative statics results. We show there how increases in \( N \) and \( I \), respectively, affect the equilibrium pricing function. Section 4 analyzes the welfare effects of an increase in \( N \). Section 5 describes the purification result and connections to the related literature. Section 6 concludes. Remaining proofs are found in the Appendix.

## 2 Model and Equilibrium

In this section, we introduce our basic model. Modifying the model by Varian (1980), we assume that a fixed number of firms compete in prices, where each firm is privately informed as to its cost of production. We then establish the existence of an equilibrium and characterize its features.

### 2.1 Model

We consider a market with a homogenous good and \( N \geq 2 \) ex ante identical firms. Let \( \theta_i \) denote the unit cost level for firm \( i \). We assume that cost draws are iid across firms. Specifically, cost type \( \theta_i \) is drawn from the support \([\underline{\theta}, \overline{\theta}]\) according to the twice-continuously differentiable distribution function, \( F(\theta) \), where \( \overline{\theta} > \theta > 0 \). The corresponding density is given by \( f(\theta) \equiv F'(\theta) \), where we assume that \( f(\theta) > 0 \) on \([\underline{\theta}, \overline{\theta}]\). Importantly, we assume that each firm \( i \) is privately informed of its unit cost level \( \theta_i \). The market has a unit mass of consumers. Throughout the present section and also Section 3, we assume that each consumer has a twice-continuously differentiable demand function \( D(p) \) that satisfies \( D(p) > 0 > D'(p) \) over the relevant range of prices \( p \).\(^5\) Consumers are split into two groups. A fraction \( I \in (0, 1) \) of consumers are informed, while the remaining fraction \( U = 1 - I \) are uninformed. An informed consumer observes firms’ prices and purchases

\(^5\)In Section 4, we also consider the setting in which consumer demand is inelastic, with \( D(p) = 1 \) for all \( p \leq r \) where \( r > \overline{\theta} \).
from the firm(s) with the lowest price, whereas an uninformed consumer does not observe
prices and randomly chooses a firm from which to purchase.

We analyze the following game: (i) firms learn their own cost types, (ii) firms simulta-
neously choose their prices, and (iii) given any price information, each consumer chooses
a firm to visit and makes desired purchases. We are interested in Perfect Bayesian Equi-
libria. To this solution concept, we add two symmetry requirements. First, we focus on
equilibria in which consumers do not condition their visitation decisions on firms’ “names”
but instead treat firms symmetrically. This means that uninformed consumers randomly
pick a firm from the set of all firms, and informed consumers randomize over all firms that
set the lowest price (if more than one such firm exists). Second, we focus on equilibria in
which firms use symmetric price strategies. Note that when firms use symmetric pricing
strategies it is indeed optimal for uninformed consumers to use a random visit strategy.
It is of course always optimal for consumers to purchase off of their demand curve,
which means that uninformed consumers randomly select from firms that offer the lowest price. In what follows, we thus embed optimal consumer behavior into our definition of firm payoffs. With this simplification, we may present our definition of
equilibrium exclusively in terms of firm strategies.

Given symmetry, a pure strategy for firm $i$ may be defined as a function $p(\theta_i)$ that
maps from $[\underline{\theta}, \overline{\theta}]$ to $\mathbb{R}_+$. Let $p(\theta_{-i})$ denote the vector of price selections made by firms
other than $i$ when their cost types are given by the $(N - 1)$-tuple $\theta_{-i}$. The market share
for firm $i$ is determined by the vector of prices selected by firm $i$ and its rivals; hence,
the market share for firm $i$ may be represented as $m(p(\theta_i), p(\theta_{-i}))$, where the function
$m$ maps from $\mathbb{R}^N_+$ to $[0, 1]$. In particular, under our symmetry requirement for consumer
behavior, the market share function for firm $i$ takes the following form: if $p(\theta_i) < p(\theta_j)$
for all $j \neq i$, then $m(p(\theta_i), p(\theta_{-i})) = 1 + \frac{\mu}{k}$; if $p(\theta_i) > p(\theta_j)$ for some $j \neq i$, then
$m(p(\theta_i), p(\theta_{-i})) = \frac{\mu}{k}$; and if firm $i$ ties with $k - 1$ other firms for the lowest price, then
$m(p(\theta_i), p(\theta_{-i})) = \frac{1}{k} + \frac{\mu}{N}$. Let the interim-stage market share for firm $i$ with cost type $\theta_i$
be represented as $\mathbb{E}_{\theta_i}[m(p(\theta_i), p(\theta_{-i}))]$.

We next define $\pi(\rho_i, \theta_i) \equiv (\rho_i - \theta_i)D(\rho_i)$ as the profit that firm $i$ would make if it
were to set the price $\rho_i$ and capture the entire unit mass of consumers. We assume that
$\pi(\rho_i, \theta_i)$ is strictly concave and twice-differentiable in $\rho_i$ and has a unique maximizer
at the monopoly price, $p^*(\theta_i) = \arg \max_{\rho_i} \pi(\rho_i, \theta_i)$. We assume further that $p^*(\overline{\theta}) > \overline{\theta}$. We now define the interim-stage profit while simplifying our notation somewhat: if
firm $i$ has cost type $\theta_i$, sets the price $p(\theta_i)$ and anticipates that its rivals employ the
strategy $p$ to determine their prices upon observing their cost types, then its interim-
stage market share is $M(p(\theta_i); p) \equiv \mathbb{E}_{\theta_{-i}}[m(p(\theta_i), p(\theta_{-i}))]$ and its interim-stage profit is

\[\text{\footnotesize It is also always optimal for consumers to purchase off of their demand curve, } D(p).\]
\[ \Pi(p(\theta_i), \theta_i; p) \equiv \pi(p(\theta_i), \theta_i)M(p(\theta_i); p) \]. We may now define an equilibrium as a pricing strategy \( p \) such that, for all \( \theta_i \in [\underline{\theta}, \overline{\theta}] \) and \( \rho_i \in \mathbb{R}_+ \), \( \Pi(p(\theta_i), \theta_i; p) \geq \Pi(\rho_i, \theta_i; p) \).

It is also convenient to write equilibrium variables and define equilibrium in direct form, ignoring subscript \( i \). If a firm with cost type \( \theta \) picks the price \( p(\hat{\theta}) \) when its rivals employ the pricing strategy \( p \) to determine their prices, then its interim-stage profit is

\[ \Pi(\hat{\theta}, \theta; p) \equiv \Pi(p(\hat{\theta}), \theta; p) = \pi(p(\hat{\theta}), \theta)M(p(\hat{\theta}); p). \]

The pricing strategy \( p \) is then an equilibrium if for any \( \theta \in [\underline{\theta}, \overline{\theta}] \), the price \( p(\theta) \) is an optimal choice for a firm with cost type \( \theta \) in comparison to all “on-schedule” and “off-schedule” deviations. Formally, for all \( \theta, p(\theta) \) must satisfy (i) the on-schedule incentive constraint,

\[ \pi(p(\theta), \theta)M(p(\theta); p) \geq \pi(p(\hat{\theta}), \theta)M(p(\hat{\theta}); p) \quad \text{for all } \hat{\theta} \neq \theta \]  

(On-IC(\( \theta \))

and (ii) the off-schedule incentive constraint,

\[ \pi(p(\theta), \theta)M(p(\theta); p) \geq \pi(\hat{\rho}, \theta)M(\hat{\rho}; p) \quad \text{for all } \hat{\rho} \notin p(\overline{\theta}, \theta)]. \]  

(Off-IC(\( \theta \))

### 2.2 Equilibrium: Existence and Characterization

Having now defined an equilibrium for the model, we turn next to establish the existence of the equilibrium and to characterize its features.

By adding two constraints, On-IC(\( \theta \)) and On-IC(\( \hat{\theta} \)), we find that \([D(p(\theta))M(p(\theta); p) - D(p(\hat{\theta}))M(p(\hat{\theta}); p)][\hat{\theta} - \theta] \geq 0\), which means that \( D(p(\theta))M(p(\theta); p) \) must be nonincreasing in \( \theta \). It then follows that \( p(\theta) \) must be nondecreasing in \( \theta \). Given the existence of informed consumers, we may further conclude that \( p(\theta) \) cannot be constant over any interval of types: by slightly decreasing its price, a firm with a type on such an interval would enjoy a discrete gain in its expected market share. We conclude that \( p(\theta) \) must be strictly increasing, and it thus follows that \( M(p(\theta); p) = \frac{U}{N} + [1 - F(\theta)]^{N-1}I \). In equilibrium, the expected market share is thus determined by the location of \( \theta \) and may be henceforth denoted as \( M(\hat{\theta}) \equiv M(p(\hat{\theta}); p) = \frac{U}{N} + [1 - F(\hat{\theta})]^{N-1}I \). Since \( M(\overline{\theta}) = \frac{U}{N} \), we may conclude that a firm with the highest-cost type has no chance of winning the informed consumers and simply selects its monopoly price, \( p(\overline{\theta}) = p^m(\overline{\theta}) \). For all other cost types, the equilibrium price is lower than the monopoly price: \( p(\theta) < p^m(\theta) \) for all \( \theta < \overline{\theta}. \)

Finally, given \( I < 1 \), it is also evident that \( p(\theta) > \theta \) and thus \( \pi(p(\theta), \theta) > 0 \) for

\[ p(\theta) > p^m(\theta) \]  

is impossible in equilibrium. For \( \theta < \overline{\theta} \), it is also impossible to have \( p(\theta) = p^m(\theta) \): a firm with cost type \( \theta \) would then deviate, since when it slightly reduces its price from \( p^m(\theta) \), the
We now establish the following existence and uniqueness result.

**Proposition 1.** There exists a unique equilibrium. In this equilibrium, \( p(\theta) \) is differentiable and strictly increasing for \( \theta \in (\underline{\theta}, \bar{\theta}) \) and satisfies

\[
p'(\theta) = -\frac{\pi(p(\theta), \theta)[\partial M(\theta)/\partial \theta]}{\pi_p(p(\theta), \theta) M(\theta)} \quad \text{and} \quad p(\bar{\theta}) = p^m(\bar{\theta}),
\]

where

\[
M(\theta) = \frac{U}{N} + [1 - F(\theta)]^{N-1} I.
\]

Notice that the equilibrium pricing function acts as a sorting mechanism: firms truthfully reveal their cost types along the upward-sloping schedule \( p(\theta) \). In the Appendix, we derive the remaining necessary features (namely, the differentiability of \( p(\theta) \) and the differential equation (1)) and also provide the sufficiency argument that \( p(\theta) \) defined in (1) constitutes an equilibrium. The intuition for this sufficiency is that \( p(\theta) \) in (1) satisfies a local optimality condition, \( \Pi_1(\hat{\theta}, \theta; p) = 0 \) for \( \hat{\theta} = \theta \), which ensures that On-IC(\( \theta \)) holds for all \( \theta \), due to the single-crossing property, \( \Pi_{12} > 0 \). Further, if \( p(\theta) \) satisfies On-IC(\( \theta \)), then it also satisfies Off-IC(\( \theta \)): given that a firm with type \( \theta < \bar{\theta} \) does not mimic \( \bar{\theta} \), it also will not select a price above \( p(\bar{\theta}) \), where \( p(\bar{\theta}) = p^m(\bar{\theta}) > p^m(\theta) \); and given that a firm with type \( \theta > \bar{\theta} \) does not mimic \( \bar{\theta} \), it also will not select a price below \( p(\bar{\theta}) \), where \( p(\bar{\theta}) < p^m(\bar{\theta}) < p^m(\theta) \). Since \( p(\bar{\theta}) < p^m(\theta) \), a firm with type \( \theta \) that captures all informed consumers will not select a price below \( p(\theta) \).

Proposition 1 is related to previous findings in the literature. Drawing on arguments by Maskin and Riley (1984), Spulber (1995) establishes the existence of a unique equilibrium for the case in which all consumers are informed, \( I = 1 \). We assume instead that \( I \in (0,1) \) and provide a related proof of equilibrium existence and uniqueness. We also note that Bagwell and Wolinsky (2002) establish equilibrium existence when \( I \in (0,1) \) for the case of inelastic demand. Finally, in their study of advertising competition, Bagwell and Lee (2010) also briefly consider a benchmark game with price competition and state the findings established in Proposition 1. We provide a complete proof here.

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8 As just argued, \( p(\bar{\theta}) = p^m(\bar{\theta}) > \bar{\theta} \) is necessary. For \( \theta \in [\underline{\theta}, \bar{\theta}) \), \( p(\theta) \leq \theta \) is impossible, since a firm with cost type \( \theta \) could earn strictly positive profit by deviating to \( p^m(\bar{\theta}) > \bar{\theta} \) (i.e., by mimicking type \( \bar{\theta} \)) and selling at least to its share of uninformed consumers, \( U/N > 0 \).

9 The single-crossing property implies that lower types find the expected-market-share increase that accompanies a price reduction more appealing than do higher types.
3 Equilibrium Pricing: Comparative Statics

In this section, we further characterize the equilibrium pricing function by performing two kinds of comparative statics. First, we investigate how an increase in the number of firms, $N$, affects the equilibrium price function, $p(\theta)$. We consider in particular how an increase in $N$ affects pricing differently for lower-cost and higher-cost firms. We also characterize $p(\theta)$ in the limit as $N$ goes to infinity. Second, we consider how an increase in the fraction of informed consumers, $I$, affects the equilibrium pricing function.

3.1 Number of Firms and Equilibrium Pricing

We begin by considering the impact of the number of firms on equilibrium pricing. A difficulty is that we do not have a closed-form expression for the equilibrium pricing function, $p(\theta)$; rather, $p(\theta)$ is characterized by the differential equation (1).

We therefore proceed in two steps. First, we use standard envelope arguments and represent interim-stage profit as follows:\(^{10}\)

$$\Pi(\theta, \theta; p) = \pi(p^m(\bar{\theta}), \bar{\theta}) \frac{U}{N} + \int_{\theta}^{\bar{\theta}} D(p(x))M(x)dx. \quad (2)$$

Notice that the interim-stage profit is strictly positive for all $\theta$ and consists of two terms: profit at the top and information rents. Intuitively, after accounting for incentive constraints, a firm of type $\theta$ can earn the profit that it would earn were it to have the highest type, $\bar{\theta}$, plus the information rents that are enjoyed as a consequence of its actual cost being lower than that of the highest type. The benefit of having a lower actual cost is greater when more units are demanded, and it thus follows that the information rents enjoyed by a firm of type $\theta$ are larger when lower prices are selected by higher types.

Our second step is to evaluate $\frac{\partial p(\theta)}{\partial N}$. To this end, we use (2) and $\Pi(\theta, \theta; p) \equiv \pi(p(\theta), \theta)M(\theta)$, and observe that the equilibrium $p(\theta)$ satisfies

$$\varphi(\theta) \equiv \pi(p(\theta), \theta)M(\theta) - \pi(p^m(\bar{\theta}), \bar{\theta}) \frac{U}{N} - \int_{\theta}^{\bar{\theta}} D(p(x))M(x)dx = 0. \quad (3)$$

For a given type $\theta < \bar{\theta}$, we next use (3) to find that

$$\frac{\partial p(\theta)}{\partial N} = -\frac{\varphi_N(\theta)}{\pi_p(p(\theta), \theta)M(\theta)}. \quad (4)$$

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\(^{10}\)See Milgrom and Segal (2002).
We now consider all types $\theta < \theta$ and check the sign of $\frac{\partial p(\theta)}{\partial N}$. Since the denominator in (4) is positive for all $\theta < \theta$, the sign of $\frac{\partial p(\theta)}{\partial N}$ is determined by the sign of $-\varphi_N(\theta)$ for all $\theta < \theta$. As we show in the Appendix, the endpoints conditions satisfy $\varphi_N(\theta) > 0$ and $\varphi_N(\theta) = 0$. Furthermore, there exists a unique type $\theta^* \in (\theta, \theta)$ and a unique type $\theta^* \in (\theta, \theta^*)$ such that $\varphi_N(\theta) > 0$ for $\theta \in [\theta, \theta^*)$, $\varphi_N(\theta) < 0$ for $\theta \in (\theta^*, \theta)$ and $\varphi_N(\theta) = 0$ for $\theta \in \{\theta^*, \theta\}$. The level of $\theta^*$ is determined by $\eta(\theta^*) = 0$ where

$$
\eta(\theta) = \frac{U}{N} \left( \frac{1}{N} + \frac{1}{N-1} \right) + \frac{U}{N} \ln [1 - F(\theta)] + \frac{I}{N-1} [1 - F(\theta)]^{N-1}. \tag{5}
$$

Observe that $\eta(\theta)$ is strictly decreasing in $\theta$, and that there exists a unique value $\theta^* \in (\theta, \theta)$ at which $\eta(\theta^*) = 0$.

We may now present the following finding:

**Proposition 2.** For any $N$, there exists a unique type $\theta^* \in (\theta, \theta^*)$ such that an increase in $N$ implies that the equilibrium $p(\theta)$ strictly decreases for $\theta \in [\theta, \theta^*)$, strictly increases for $\theta \in (\theta^*, \theta)$ and is unchanged for $\theta \in \{\theta^*, \theta\}$.

An interesting implication of Proposition 2 is that the potential extent of price dispersion, as measured by the support of possible equilibrium prices, $[p(\theta), p^m(\theta)]$, is larger in markets with a greater number of firms.

At a broad level, we may understand Proposition 2 as indicating that an increase in $N$ generates price-decreasing and price-increasing forces. An increase in $N$ may lead to lower prices, as some firms compete more aggressively for informed consumers, but it may also lead to higher prices, as some firms become discouraged about the prospect of winning informed consumers and thus focus on sales to uninformed consumers. Proposition 2 reveals that the balance of these competing forces hinges on a firm’s cost type: as $N$ rises, lower-cost firms compete more aggressively and reduce prices, while higher-cost firms perceive a reduced chance of winning informed consumers and increase prices.

At a more specific level, we may provide further understanding by describing the structure of the proof. In the Appendix, we show first that the price of the highest type, $\theta$, is unaltered by an increase in $N$ (i.e., $\varphi_N(\theta) = 0$). We then consider the lowest type, $\theta$, and suppose to the contrary that the price of this type weakly increases when $N$ rises. Under this supposition, we show that each higher type, $\theta \in (\theta, \theta)$, must weakly increase its price as well. Given downward-sloping demand, it then follows that information rents for type $\theta$ must strictly fall when $N$ is increased; in fact, by (3), we may then conclude that $\pi(p(\theta), \theta)$ must strictly fall when $N$ is increased.\(^{11}\) This conclusion, however, implies

\(^{11}\) Notice that $M(\theta)$ strictly decreases in $N$ for all $\theta$. Given $M(\theta) = \frac{U}{N} + I$, we may also confirm that
that an increase in \( N \) leads to a strict reduction in \( p(\bar{\theta}) \), which contradicts our original supposition. Thus, an increase in \( N \) must induce a strictly lower price for the lowest type, \( \bar{\theta} \) (i.e., \( \varphi_N(\bar{\theta}) > 0 \)). The remaining arguments in the proof then establish that there is a unique type \( \theta^* \in (\bar{\theta}, \bar{\theta}) \) whose price is unchanged when \( N \) increases and that \( \theta^* < \theta^{**} \).

We next characterize the equilibrium in the limit where \( N \) is sufficiently large. As \( N \) goes to infinity, the prospect of winning informed consumers approaches zero for a firm with any cost type \( \theta > \bar{\theta} \); thus, in the limit, the range over which the price-increasing force dominates the price-decreasing force expands to all \( \theta > \bar{\theta} \). Formally, it follows from (5) that the level of \( \theta^{**} \) is strictly decreasing in \( N \), with \( \theta^{**} \to \bar{\theta} \) and thus \( \theta^* \to \bar{\theta} \) as \( N \to \infty \).\(^{12}\) For further characterization, we use the interim-stage profit in (2):

\[
\pi(p(\theta), \theta) = \pi(p^m(\bar{\theta}), \bar{\theta}) \frac{U/N}{M(\theta)} + \int_{\theta}^{\bar{\theta}} D(p(x)) \frac{M(x)}{M(\theta)} dx.
\]

For \( \theta = \theta \), since \( M(\theta) = \frac{U}{N} + I \), if \( N \to \infty \), then the RHS approaches zero and thus \( p(\bar{\theta}) \to \bar{\theta} \). For \( \theta \in (\bar{\theta}, \bar{\theta}) \), if \( N \to \infty \), then \( p(\theta) \) approaches \( \tilde{p}(\theta) \) that satisfies

\[
\pi(\tilde{p}(\theta), \theta) = \pi(\tilde{p}(\bar{\theta}), \bar{\theta}) + \int_{\theta}^{\bar{\theta}} D(\tilde{p}(x)) dx,
\]

where \( \tilde{p}(\bar{\theta}) = p^m(\bar{\theta}) \). To derive the RHS of (7), we refer to (6) and observe that, for \( x \in (\theta, \bar{\theta}) \), the term \( \frac{M(x)}{M(\theta)} \) < 1 and approaches 1 as \( N \to \infty \):

\[
\lim_{N \to \infty} \frac{M(x)}{M(\theta)} = \lim_{N \to \infty} U + N[1 - F(x)]^{N-1} = 1.
\]

Similarly, \( \frac{U/N}{M(\theta)} \) < 1 and approaches 1 as \( N \to \infty \).\(^{13}\) Now, from (7), we have

\[
- \int_{\theta}^{\bar{\theta}} \frac{d}{dx} \pi(\tilde{p}(x), x) dx = \int_{\theta}^{\bar{\theta}} D(\tilde{p}(x)) dx.
\]

We next follow two steps and identify the “limit price” \( \tilde{p}(\theta) \) for \( \theta \in (\bar{\theta}, \bar{\theta}) \). First, for \( \frac{U/N}{M(\theta)} \) and \( \frac{M(x)}{M(\theta)} \) are strictly decreasing in \( N \) for \( x > \bar{\theta} \).

\(^{12}\) The proof for this result is found in the proof of Proposition 3.

\(^{13}\) The terms \( N[1 - F(\theta)]^{N-1} \) and \( N[1 - F(x)]^{N-1} \) increase in \( N \) for small \( N \), but decrease in \( N \) for large \( N \). In the limit, these terms approach zero: for \( \theta \in (\bar{\theta}, \bar{\theta}) \), \( \lim_{N \to \infty} N[1 - F(\theta)]^{N-1} \) becomes

\[
\lim_{N \to \infty} \frac{N}{\left[1 - F(\theta)\right]^{N-1}} = \lim_{N \to \infty} \frac{1}{\left[1 - F(\theta)\right]^{N-1} \ln \frac{1}{1 - F(\theta)}} = 0.
\]
\( \theta \in (\bar{\theta}, \bar{\theta}) \), we have \( \bar{p}(\theta) \leq p^m(\theta) \). This result follows since, as previously noted, for any given \( N \), \( p(\theta) < p^m(\theta) \) for \( \theta < \bar{\theta} \). Second, for \( \theta \in (\bar{\theta}, \bar{\theta}) \), if \( N \) becomes sufficiently large, then \( \text{On-IC}(\theta) \) is satisfied only if \( p(\theta) \) approaches either the monopoly price \( p^m(\theta) \) or the fixed price \( p^m(\bar{\theta}) \), since \( \text{On-IC}(\theta) \) implies

\[
\pi(p(\theta), \theta) (U + N[1 - F(\theta)]^{N-1}I) \geq \pi(\bar{p}(\theta), \theta) \left( U + N[1 - F(\bar{\theta})]^{N-1}I \right)
\]

for all \( \bar{\theta} > \theta \),

where the terms \( N[1 - F(\theta)]^{N-1}I \) and \( N[1 - F(\bar{\theta})]^{N-1}I \) approach zero in the limit.\(^{14}\) Since \( p^m(\theta) < p^m(\bar{\theta}) \) for \( \theta < \bar{\theta} \), these two steps lead to \( \bar{p}(\theta) = p^m(\theta) \) for \( \theta \in (\bar{\theta}, \bar{\theta}) \). We note that (8) is satisfied by this limit price.

**Proposition 3.** In the limit where \( N \to \infty \), (i) \( \theta^* \to \bar{\theta} \), (ii) \( p(\theta) \to \bar{\theta} \), and for \( \theta \in (\bar{\theta}, \bar{\theta}) \), \( p(\theta) \to p^m(\theta) \).

Proposition 3 captures a simple idea: in the limit, the prospect of winning informed consumers approaches zero for all \( \theta \in (\bar{\theta}, \bar{\theta}) \), and hence the price \( p(\theta) \) approaches the monopoly price for all \( \theta \in (\bar{\theta}, \bar{\theta}) \). This result may be broadly regarded as an incomplete-information confirmation of the limit results reported in Rosenthal (1980).\(^{15}\)

### 3.2 Informed Consumers and Equilibrium Pricing

We now consider the impact of the fraction of informed consumers, \( I \), on equilibrium pricing. Referring to (1) in Proposition 1, we see that \( p(\bar{\theta}) = p^m(\bar{\theta}) \) is independent of \( I \). By contrast, for \( \theta < \bar{\theta} \), (1) implies that \( p'(\theta) \) strictly increases with \( I \), since

\[
\frac{d}{dI} \frac{\partial M(\theta)}{\partial \theta} = -\frac{(N - 1)(1 - F(\theta))^{N-2}f(\theta)}{N[1 - F(\theta)]^{N-1}I^2} < 0.
\]

These observations establish the following result:

**Proposition 4.** If \( I \) increases, then the equilibrium \( p(\theta) \) strictly decreases for \( \theta \in [\bar{\theta}, \bar{\theta}) \) and is unchanged for \( \theta = \bar{\theta} \).

For a given number of firms, a larger share of informed consumers thus leads to a strictly lower equilibrium price for each cost type other than the highest cost-type, whose

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\(^{14}\)In the limit, a firm with \( \theta \in (\bar{\theta}, \bar{\theta}) \) will not mimic type \( \bar{\theta} \) since \( p(\bar{\theta}) \to \bar{\theta} \).

\(^{15}\)As discussed further in Section 5, Rosenthal (1980) analyzes the symmetric mixed-strategy equilibria of a complete-information model.
price is unaffected.\footnote{Limiting results may also be reported. As $I \to 0$ (i.e., $U \to 1$), the equilibrium pricing function approaches the monopoly pricing function. Formally, if $I \to 0$, then for all $\theta \in [\underline{\theta}, \overline{\theta}]$, $\frac{U/N}{M(\theta)} \to 1$ and $\frac{M(\theta)}{M(N)} \to 1$ in (6) and hence, $p(\theta) \to p^m(\theta)$. Thus, monopoly pricing occurs as a limiting result when the number of firms goes to infinity (see Propostion 3) or when the share of informed consumers goes to zero. At the other extreme, if $I = 1$ as assumed in Spulber (1995), then the price at the top becomes $p(\overline{\theta}) = \overline{\theta}$, and $\eta(\theta)$ in (5) becomes $\eta(\theta) = \frac{1}{N-1} [1 - F(\theta)]^{N-1}$, which indicates that $\varphi_N(\theta) > 0$ for all $\theta < \overline{\theta}$ given $\varphi_N(\overline{\theta}) = 0$; thus, $\frac{\partial p(\theta)}{\partial N} < 0$ for all $\theta < \overline{\theta}$. Note that the difference between $I = 1$ and $I = 1 - \varepsilon$ causes a discontinuity in the choice of $p(\theta)$ due to a shift in the boundary value (for $I = 1 - \varepsilon$, a firm with type $\overline{\theta}$ selects $p(\overline{\theta}) = p^m(\overline{\theta})$ no matter how small is $\varepsilon > 0$).} Since the equilibrium price function drops pointwise for all $\theta < \overline{\theta}$, it follows immediately that an increase in the share of informed consumers delivers strictly higher expected welfare for both informed and uninformed consumers. We note also that an increase in the share of informed consumers leads to a wider possible dispersion of equilibrium prices, since the lowest possible price, $p(\overline{\theta})$, strictly falls while the highest possible price, $p(\overline{\theta})$, is unaffected. It is interesting to contrast these findings with those reported in Proposition 2 concerning the impact of an exogenous increase in the number of firms. An increase in the number of firms is similar to an increase in the share of informed consumers in that both changes lead to a wider possible dispersion of equilibrium prices; however, an increase in the number of firms is different than an increase in the share of informed consumers in that the former leads to lower equilibrium prices only for lower-cost firms while the latter leads to lower equilibrium prices for firms with all cost types other than the maximal cost type. The latter point indicates that the implication of an increase in the number of firms for the average price and thus uninformed consumer welfare may be complex. We explore this implication further in the next section.

4 Number of Firms and Welfare

In this section, we investigate how an increase in $N$ affects welfare. Specifically, we consider the effects of an increase in $N$ on consumer surplus, producer surplus and total surplus, and we also explore the distinct effects of an increase in $N$ on the welfares of informed and uninformed consumers.

An uninformed consumer faces the expected (posted) price:

$$P^U = \mathbb{E}_\theta p(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} p(\theta) dF(\theta).$$

An informed consumer observes prices and faces the expected minimum price in the
market:

\[ P_T = \mathbb{E}_\theta p(\theta_{\min}) = \int_{\theta_0}^{\theta_{\max}} p(\theta) dG(\theta) \]

where \( \theta_{\min} \equiv \min\{\theta_1, ..., \theta_N\} \) and \( G(\theta) \equiv 1 - [1 - F(\theta)]^N \) is the distribution function of the lowest cost type among \( N \) samples. The expected transaction price is

\[ P_T = \int_{\theta_0}^{\theta_{\max}} p(\theta) dH(\theta) \]

where \( H(\theta) \equiv U \cdot F(\theta) + I \cdot G(\theta) \). Let \( \kappa \equiv \int_{p^m(\bar{\theta})}^{r} D(x) dx \) where \( r \equiv \sup\{p : D(p) > 0\} \).

We may represent the aggregate consumer surplus as

\[ CS = \int_{\theta_0}^{\theta_{\max}} \int_{p(\theta)}^{p^m(\bar{\theta})} D(x) dx dH(\theta) + \kappa. \tag{9} \]

We find it difficult to determine whether \( CS \) increases in \( N \) directly from (9). To further characterize the determinants of the relationship between \( CS \) and \( N \), we therefore proceed with the following three steps. First, using \( p(\bar{\theta}) = p^m(\bar{\theta}) \), we integrate by parts and rewrite (9) as

\[ CS = \int_{\theta_0}^{\theta_{\max}} H(\theta) D(p(\theta)) p'(\theta) d\theta + \kappa. \tag{10} \]

Second, using the interim-stage profit (2), we integrate by parts and represent expected profit as

\[ \int_{\theta_0}^{\theta_{\max}} \Pi(\theta, \theta; p) dF(\theta) = \frac{\pi(p^m(\bar{\theta}), \theta)}{N} \frac{U}{N} + \int_{\theta_0}^{\theta_{\max}} M(\theta) D(p(\theta)) \sigma(\theta) dF(\theta), \tag{11} \]

where \( \sigma(\theta) \equiv \frac{E}{F}(\theta) \). Notice that expected profit as captured in (11) consists of profit at the top and an expected information rent term.

Our third step is to rewrite (11) in relation to \( CS \) in (10). After multiplying by \( N \), the LHS of (11) becomes

\[ N \int_{\theta_0}^{\theta_{\max}} \pi(p(\theta), \theta) M(\theta) dF(\theta) = \int_{\theta_0}^{\theta_{\max}} \pi(p(\theta), \theta) dH(\theta), \]

where the equality follows since \( N \cdot M(\theta) dF(\theta) = [U \cdot f(\theta) + I \cdot N[1 - F(\theta)]^{N-1} f(\theta)] d\theta = \)
Integrating by parts, we have
\[ \int_{\theta}^{\bar{\theta}} \pi(p(\theta), \theta) dH(\theta) = \pi(p^m(\bar{\theta}), \bar{\theta}) - \int_{\theta}^{\bar{\theta}} H(\theta) D(p(\theta)) p'(\theta) d\theta \]
\[ + \int_{\theta}^{\bar{\theta}} H(\theta) D(p(\theta)) d\theta - \int_{\theta}^{\bar{\theta}} H(\theta) [p(\theta) - \theta] D'(p(\theta)) p'(\theta) d\theta. \]

Observe that the second term on the RHS of (12) equals \( \kappa - CS \). Similarly, we rewrite the RHS of (11) after multiplying by \( N \). The second term on the RHS of (11) then takes the form
\[ N \int_{\theta}^{\bar{\theta}} M(\theta) D(p(\theta)) \sigma(\theta) dF(\theta), \]
which becomes
\[ \int_{\theta}^{\bar{\theta}} D(p(\theta)) \sigma(\theta) dH(\theta) = D(p^m(\bar{\theta})) \sigma(\bar{\theta}) - \int_{\theta}^{\bar{\theta}} H(\theta) D(p(\theta)) \sigma'(\theta) d\theta \]
\[ - \int_{\theta}^{\bar{\theta}} H(\theta) \sigma(\theta) D'(p(\theta)) p'(\theta) d\theta. \]

Rearranging both sides, we find that the equality (11) is equivalent to
\[ CS = \kappa + \pi(p^m(\bar{\theta}), \bar{\theta}) I - D(p^m(\bar{\theta})) \sigma(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} H(\theta) D(p(\theta)) [1 + \sigma'(\theta)] d\theta \]
\[ - \int_{\theta}^{\bar{\theta}} H(\theta) D'(p(\theta)) p'(\theta) [p(\theta) - \theta - \sigma(\theta)] d\theta. \]

We observe that the RHS of (14) depends on \( N \) through the equilibrium pricing function, \( p(\theta) \), and the function \( H(\theta) \). As noted in Proposition 2, whether an increase in \( N \) raises or lowers \( p(\theta) \) depends on \( \theta \). By contrast, \( H(\theta) \) is strictly increasing in \( N \) for all \( \theta \in (\theta, \bar{\theta}) \). In any case, without further structure on the demand and distribution functions, it is not clear from (14) whether \( CS \) increases or decreases in \( N \).

The expression presented in (14), however, does direct attention to one important set of circumstances under which \( CS \) unambiguously increases in \( N \). In particular, suppose first that demand is inelastic: \( D(p) = 1 \) for all \( p \leq r \) and \( D(p) = 0 \) for all \( p > r \) where \( r > \bar{\theta} \). For the case of inelastic demand, \( p^m(\theta) = r \) for all \( \theta \) and \( \kappa = 0 \). We note that Propositions 1-4 hold as well when demand is inelastic. Suppose second that \( 1 + \sigma'(\theta) > 0 \) for all \( \theta \). To motivate this assumption, we note that for many popular distributions, \( F \) is log-concave (\( F\) is nondecreasing in \( \theta \)) and thus \( \sigma'(\theta) \geq 0 \) holds for all \( \theta \).

Referring to (14) and recalling that \( H(\theta) \) is strictly increasing in \( N \) for all \( \theta \in (\theta, \bar{\theta}) \), we see that

\[ \text{Log-concavity of } F \text{ is commonly assumed in the contract literature and is satisfied by many distribution functions.} \]
an increase in $N$ strictly raises aggregate consumer surplus if demand is inelastic and $1 + \sigma'(\theta) > 0$ for all $\theta$.

Indeed, if demand is inelastic, then the differential equation (1) is immediately solved from the interim-stage profit (2) and has the analytical solution:

$$p(\theta) = \theta + \pi(r, \theta) \frac{U/N}{M(\theta)} + \int_{\theta}^{\overline{\theta}} M(x) \frac{d}{d\theta}.$$  \hfill (15)

Plugging (15) into $P^T = \int_{\theta}^{\overline{\theta}} p(\theta)dH(\theta) = N \int_{\theta}^{\overline{\theta}} p(\theta)M(\theta)dF(\theta)$, we find that

$$P^T = rU + I \int_{\theta}^{\overline{\theta}} \theta dG(\theta) + I \int_{\theta}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} N(1 - F(x))^{N-1}dxdF(\theta).$$

After successive integration by parts, the third term may be re-written as

$$I \int_{\theta}^{\overline{\theta}} \int_{\theta}^{\overline{\theta}} N(1 - F(x))^{N-1}dxdF(\theta) = I \int_{\theta}^{\overline{\theta}} \sigma(\theta)N(1 - F(\theta))^{N-1}dF(\theta)$$

$$= I \int_{\theta}^{\overline{\theta}} (1 - F(\theta))^{N-1} \sigma'(\theta)d\theta$$

Similarly, the second term takes the form $I \int_{\theta}^{\overline{\theta}} \theta dG(\theta) = I[\overline{\theta} - \int_{\theta}^{\overline{\theta}} G(\theta)d\theta]$. Substituting these expressions and simplifying, we represent the expected transaction price as

$$P^T = r - \pi(r, \overline{\theta})I - \int_{\theta}^{\overline{\theta}} I[G(\theta) - (1 - F(\theta))^{N-1}\sigma'(\theta)]d\theta$$

$$= r - \pi(r, \overline{\theta})I + \sigma(\overline{\theta})I - \int_{\theta}^{\overline{\theta}} [H(\theta) - UF(\theta)][1 + \sigma'(\theta)]d\theta.$$ \hfill (16)

Given that $H(\theta)$ strictly increases in $N$ for all $\theta \in (\theta, \overline{\theta})$, if $1 + \sigma'(\theta) > 0$ for all $\theta$, then $P^T$ strictly decreases in $N$. Since $CS = r - P^T$, this finding matches our finding from (14) that $CS$ strictly increases in $N$ when demand is inelastic and $1 + \sigma'(\theta) > 0$ for all $\theta$.

For the case of inelastic demand, limiting results take a particularly simple form. Using (15), it is straightforward to verify Proposition 3 for the inelastic demand setting: as $N \to \infty$, $p(\theta) \to r \equiv p^m(\theta)$ for all $\theta \in (\theta, \overline{\theta})$ and $p(\theta) \to \overline{\theta}$. The corresponding implication for the limiting transaction price is delivered using (16): when demand is inelastic, as $N \to \infty$, $H(\theta) - UF(\theta) \to I$ for all $\theta > \theta$, and so $P^T \to \overline{\theta}I + rU$. Intuitively, as the number of firms goes to infinity, competition among firms with the lowest possible cost type implies that informed consumers receive the price $\overline{\theta}$ whereas uniformed consumers
face “discouraged” firms and receive the monopoly price, \( r \).

We next consider aggregate producer surplus, represented by

\[
PS = N \cdot \int_{\theta} \Pi(\theta; \theta; p) dF(\theta).
\]

From (11), we have

\[
PS = \pi(p^m(\theta), \theta) U + \int_{\theta} D(p(\theta)) \sigma(\theta) dH(\theta). \tag{17}
\]

Using the rearrangement in (13), we may now conclude that \( PS \) strictly decreases in \( N \) if demand is inelastic and \( F \) is log-concave (so that \( \sigma'(\theta) \geq 0 \) for all \( \theta \)).

Finally, we may now use (13), (14) and (17) to derive total surplus, represented by \( TS = CS + PS \), as

\[
TS = \kappa + \pi(p^m(\theta), \theta) + \int_{\theta} H(\theta) D(p(\theta)) \left( 1 - [p(\theta) - \theta] \frac{D'(p(\theta))p'(\theta)}{D(p(\theta))} \right) d\theta. \tag{18}
\]

Observe that, if demand is inelastic, then \( TS \) as represented in (18) strictly increases in \( N \). Intuitively, when more firms enter, the expected cost of the firm that sells to informed consumers decreases. This result holds for any distribution function, \( F \).

We now summarize our welfare analysis to this point:

**Proposition 5.** Suppose that demand is inelastic. (i) If \( 1 + \sigma'(\theta) > 0 \) for all \( \theta \), then an increase in \( N \) strictly raises \( CS \); and if \( F \) is log-concave, then an increase in \( N \) strictly lowers \( PS \). (ii) An increase in \( N \) strictly raises \( TS \) for any \( F \).

Under inelastic demand and \( 1 + \sigma'(\theta) > 0 \), Proposition 5 implies that, whenever uninformed consumers suffer from an increase in \( N \), informed consumers benefit from it.

We next consider the welfare effects of an increase in \( N \) on the two respective groups of consumers. Let \( CS^U \) and \( CS^I \) represent surplus of uninformed and informed consumers, respectively. Integrating (6), we have the following expected profit expression:

\[
\int_{\theta} [p(\theta) - \theta] D(p(\theta)) dF(\theta) = \pi(p^m(\theta), \theta) \int_{\theta} \frac{U/N}{M(\theta)} dF(\theta) + \int_{\theta} \int_{\theta} D(p(x)) \frac{M(x)}{M(\theta)} dx dF(\theta).
\]

Next, arguing as in (9) and (10), we may derive that \( CS^U = \kappa + \int_{\theta} F(\theta) D(p(\theta)) p'(\theta) d\theta. \)

Integrating by parts the LHS of the expected profit expression and substituting in the following.

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18 Note that \( F \) must exhibit strict log-concavity, \( \sigma'(\theta) > 0 \), for a range of \( \theta \) that is adjacent to \( \theta \).

19 Our welfare findings here also hold for sufficiently inelastic linear demand functions.
derived expression for $CS^U$, we find that

$$CS^U = \kappa + \pi(p^m(\theta), \bar{\theta}) + \int_{\theta}^{\bar{\theta}} F(\theta)D(p(\theta))d\theta - \int_{\theta}^{\bar{\theta}} F(\theta)[p(\theta) - \theta]D'(p(\theta))p'(\theta)d\theta$$

(19)

$$- \pi(p^m(\theta), \bar{\theta}) \int_{\theta}^{\bar{\theta}} \frac{U/N}{M(\theta)}dF(\theta) - \int_{\theta}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} D(p(x)) \frac{M(x)}{M(\theta)} dx dF(\theta).$$

Without further structure, it is challenging to use (19) to determine the impact of an increase in $CS^U$.

We thus turn again to the case of inelastic demand. Evaluating $CS^U$ in (19) when $D = 1, D' = 0, p^m(\theta) = r$ and $\kappa = 0$, and letting $P^U = r - CS^U$, we get that

$$P^U = r - \pi(r, \bar{\theta}) - \int_{\theta}^{\bar{\theta}} F(\theta)d\theta + \pi(r, \bar{\theta}) \int_{\theta}^{\bar{\theta}} \frac{U/N}{M(\theta)}dF(\theta) + \int_{\theta}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} \frac{M(x)}{M(\theta)} dx dF(\theta).$$

(20)

Thus, even when demand is inelastic, it is not clear whether an increase in $N$ raises or lowers $CS^U$. The effect of an increase in $N$ operates through the last two terms of the RHS of (20). As we confirm in the Appendix, we can show that the term $\int_{\theta}^{\bar{\theta}} \frac{U/N}{M(\theta)}dF(\theta)$ is independent of the support, $\bar{\theta} - \theta$, and is strictly increasing in $N$, by rewriting the term as

$$\int_{\theta}^{\bar{\theta}} \frac{U/N}{M(\theta)}dF(\theta) = \int_{0}^{1} \frac{1}{1 + \frac{1}{N}xN^{-1}}dx$$

for any $F$, (21)

and showing that $\int_{0}^{1} \frac{1}{1 + \frac{1}{N}xN^{-1}}dx$ is strictly increasing in $N$. By contrast, the effect of an increase in $N$ on the term $\int_{\theta}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} \frac{M(x)}{M(\theta)} dx dF(\theta)$ is elusive and depends on the level of $N$ and the characteristics of $F$.

We can report, however, specific circumstances in which a larger number of firms is disadvantageous to uninformed consumers. In line with our earlier discussion of limiting results, it follows easily from (20) that $P^U$ approaches $r$ as $N$ goes to infinity; thus, for any given $N$, there exists $N^* > N$ such that $CS^U$ is strictly higher with $N$ firms than when the number of firms exceeds $N^*$. As well, we can identify a setting in which $P^U$ as represented in (20) strictly increases in $N$. In particular, we find that an increase in $N$}

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20When demand is inelastic, we could equivalently derive $P^U$ by substituting (15) into $\int_{\theta}^{\bar{\theta}} p(\theta)dF(\theta)$.

21In the proof of Proposition 6 in the Appendix, we directly build on the complete-information analysis by Morgan, Orzen and Sefton (2006) and show that $\frac{\partial}{\partial N} \int_{0}^{1} \frac{1}{1 + \frac{1}{N}xN^{-1}}dx > 0$ holds. As we discuss below and confirm in the Appendix, the model analyzed in this paper is a puriﬁed version of the complete-information model when the support $\bar{\theta} - \theta$ becomes sufﬁciently small. Janssen and Moraga-Gonzalez (2004) show that $\int_{0}^{1} \frac{1}{1 + \frac{1}{N}xN^{-1}}dx$ is strictly increasing in $N$ when $N$ is integer. The current result is based on the differentiability in $N$. 

18
has a vanishing effect on the term \( \int_\theta^\bar{\theta} \int_{\theta}^{M(x)} M(x) dF(\theta) \) when the support of possible cost types, \( \bar{\theta} - \theta \), becomes sufficiently small. Combining these findings with our results about the term isolated in (21), we can thus conclude that \( P^U \) strictly increases in \( N \) when \( \bar{\theta} - \theta \) is sufficiently small:

The following proposition contains our main findings about the effects of an increase in \( N \) on uninformed and informed consumers, respectively:

**Proposition 6.** Suppose that demand is inelastic. (i) For any \( N \), there exists \( N^* > N \) such that \( CS^U \) is strictly higher and \( CS^I \) is strictly lower with \( N \) firms than for any number of firms in excess of \( N^* \). (ii) For \( \bar{\theta} - \theta \) sufficiently small, an increase in \( N \) strictly lowers \( CS^U \) for any \( N \). (iii) For \( \bar{\theta} - \theta \) sufficiently small, an increase in \( N \) strictly increases \( CS^I \) for any \( N \).

Under inelastic demand, Proposition 6 identifies circumstances under which uninformed consumers are harmed, and informed consumers are helped, when the number of firms is larger.\(^{22}\) For uninformed consumers, part (i) follows from our discussion above, and we prove part (ii) in the Appendix. We confirm there that the term \( \int_\theta^\bar{\theta} ^{\bar{\theta}/N} M(x) dF(\theta) \) is independent of \( \bar{\theta} - \theta \) and strictly increasing in \( N \), and we also show there that the term \( \frac{\partial}{\partial N} \int_\theta^\bar{\theta} \int_{\theta}^{M(x)} M(x) dF(\theta) \) approaches zero as \( \bar{\theta} - \theta \) approaches zero. For informed consumers, part (i) follows since as noted above \( P^I \to \theta \) as \( N \to \infty \). Finally, in the Appendix, we show that \( \frac{\partial P^U}{\partial N} \) approaches zero as \( \bar{\theta} - \theta \) goes to zero. With this finding in place, we can conclude that part (iii) follows from part (ii).

For the remaining case in which \( N \) may be small or \( \bar{\theta} - \theta \) may be large, we present numerical examples, using the analytical solution (15) under inelastic demand and using uniform and “truncated” normal distributions of cost types.\(^{23}\) In the Appendix, Table 1 selects a few results from extensive examples. The following remark summarizes the features found in our numerical work:

**Remark.** Suppose that demand is inelastic. In our numerical work, the following effects are observed: (i) \( \frac{\partial P^I}{\partial N} < 0 \) for any \( N \geq 2 \). (ii) If the heterogeneity of cost types is sufficiently large, then there exists \( N^* > 2 \) such that \( \frac{\partial P^U}{\partial N} < 0 \) for \( N < N^* \) and \( \frac{\partial P^U}{\partial N} > 0 \) for \( N > N^* \). Otherwise, \( \frac{\partial P^U}{\partial N} > 0 \) for any \( N \geq 2 \).

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\(^{22}\)See Morgan, Orzen and Sefton (2006) for related findings in the limiting case of complete information.

\(^{23}\)The density function of a normal random variable with mean and variance, \( \mu \) and \( \sigma^2 \), is given by \( \lambda(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{\theta - \mu}{\sigma}\right)^2} \) where \( -\infty < \theta < \infty \). The distribution function is \( \Lambda(\theta) = \int_{-\infty}^{\theta} \lambda(x) dx \). The density function under a truncated normal distribution is defined as \( f(\theta) = \frac{\lambda(\theta)}{\Lambda(\bar{\theta}) - \Lambda(\theta)} \) if \( \theta \leq \theta \leq \bar{\theta} \), and \( f(\theta) = 0 \) otherwise. The associated distribution function is \( F(\theta) = \int_{-\infty}^{\theta} f(x) dx \).
We say that the heterogeneity of cost types is sufficiently large when the support of possible cost types, $\bar{\theta} - \underline{\theta}$, is sufficiently large, and cost types are sufficiently dispersed as in the uniform distribution, or in the normal distribution with large variance. Under these conditions in our numerical work, there exists $N^*$ below (above) which an increase in $N$ benefits (harms) uninformed consumers. Otherwise, an increase in $N$ harms uninformed consumers for any $N \geq 2$. Regardless of these conditions, an increase in $N$ benefits informed consumers.

5 Purification and Related Literature

In this section, we explain that the price equilibrium stated in Proposition 1 can be understood as a purification of the mixed-strategy equilibrium for a complete-information benchmark model, where production costs are fixed at a constant $c > 0$. Building on this result, we then further describe the relationship of our findings to those in the literature that follows Varian’s (1980) analysis.

Consider an incomplete-information game, in which unit production costs, $c(\theta)$, strictly rise with type, $\theta$. In the Appendix, we establish the following purification result: if a firm with type $\theta$ uses the unique pricing equilibrium $p(\theta)$ of the incomplete-information game, then the probability distribution induced by $p$ is approximately the same as the distribution of prices in the mixed-strategy equilibrium of the complete-information game, when $c(\theta)$ approximates the constant $c$ over the support $[\underline{\theta}, \bar{\theta}]$. This result offers a useful link between the complete- and incomplete-information games: the complete-information setting corresponds to the limiting case in which the support of possible costs, $c(\theta) - c(\underline{\theta})$, is sufficiently small.

Building on this result, we may clarify the relationship between the predictions of our model and those that emerge from Varian’s (1980) model, in which information is complete and demand is inelastic. Specifically, our comparison is with Varian’s (1980) model when the number of firms is taken as exogenous. Varian (1980) endogenizes the number of firms with a free-entry condition. See also footnote 1 above.

In Varian’s (1980) model, an increase in $N$ has no effect on $CS$, $PS$ and $TS$. The counterpart result in our model occurs when demand is inelastic and $\bar{\theta} - \underline{\theta}$ approaches zero. In that case, the effects that we describe in Proposition

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24 This analysis builds on Bagwell and Wolinsky (2002) with a slight extension from their use of inelastic demand to consider downward-sloping demand functions.

25 Specifically, our comparison is with Varian’s (1980) model when the number of firms is taken as exogenous. Varian (1980) endogenizes the number of firms with a free-entry condition. See also footnote 1 above.

26 In the complete-information benchmark model, we find that, for any demand function, $PS$ is constant in $N$. In the same model, if demand is inelastic, then $TS$ is constant in $N$ and so $CS$ is constant in $N$. For the complete-information model, we also numerically confirmed that, for a family of downward-sloping demand functions, an increase in $N$ lowers $CS^U$, raises $CS^I$ and lowers $CS$ and $TS$. 

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5 vanish: an increase in $N$ affects $CS$, $PS$ and $TS$ only through terms that vanish as $\bar{\theta} - \theta$ goes to zero. Morgan, Orzen and Sefton (2006) study Varian’s (1980) model and find that an increase in $N$ lowers $CS^U$ and raises $CS^I$. The counterpart result in our model again arises when demand is inelastic and as $\bar{\theta} - \theta$ approaches zero. Under these conditions, $P^U$ and $P^I$ approach the associated complete-information values. From this perspective, our findings in Proposition 6 (ii) and (iii) provide a natural pure-strategy counterpart to Morgan, Orzen and Sefton’s (2006) findings. Proposition 6 (i) then establishes a related finding when the number of firms is increased to a sufficiently large number, even when the support of possible cost types may be large so that the purification result does not apply. Our numerical findings provide further insights for other cases. Rosenthal’s (1980) work is also related. He studies a complete-information model and reports that consumers may be harmed in the resulting mixed-strategy equilibrium when the number of firms increases. Contrary to our model, however, his model assumes that the number of uninformed consumers per firm is invariant with respect to the number of firms.

Second, we note that price dispersion persists in our model even when $\bar{\theta} - \theta$ becomes small. Using the purification result, we can also relate this finding to the earlier literature. Specifically, if $\bar{\theta} - \theta$ approaches zero, the support of equilibrium prices, $[p(\theta), p^m(\bar{\theta})]$, approximates the support of the mixed-strategy equilibrium that exists in the complete-information benchmark model. Observe that, given the price at the top $p^m(\bar{\theta})$, the price at the bottom $p(\theta)$ is determined by

$$\pi(p(\theta), \theta) = \pi(p^m(\bar{\theta}), \bar{\theta}) \frac{U/N}{M(\bar{\theta})} + \int_{\theta}^{\bar{\theta}} D(p(x)) \frac{M(x)}{M(\theta)} dx,$$

where the last term goes to zero as $\bar{\theta} - \theta$ approaches zero since $D(p(\theta)) \int_{\theta}^{\bar{\theta}} \frac{D(p(x))}{D(p(\bar{\theta}))} \frac{M(x)}{M(\theta)} dx < D(p(\theta))(\bar{\theta} - \theta)$. Further, under inelastic demand, if $\bar{\theta} - \theta$ approaches zero, then the expected value and variance of equilibrium prices approach those found in Varian’s (1980) model.

More generally, while the purification result clarifies the relationships between our predictions and those that emerge from Varian’s (1980) model, we emphasize that our analysis considers a more general setting in which demand may be downward sloping and the support of possible costs may be large so that the purification result does not apply. The more general setting facilitates new results, such as the finding in Proposition 2 that an increase in the number of firms affects differentially the pricing of lower- and higher-cost firms, and directs attention to the assumptions on the model and distribution function that suffice for the generalization of other results.
6 Conclusion

The relationship between the number of firms and price competition is a central issue in economics. To explore this relationship, we modify Varian’s (1980) model and assume that firms are privately informed about their costs of production. Allowing that the support of possible cost types may be large, we show that an increase in the number of firms induces lower (higher) prices for lower-cost (higher-cost) firms. We also characterize the pricing distribution as the number of firms approaches infinity, finding that the equilibrium pricing function converges to the monopoly pricing function for all but the lowest possible cost type. If demand is inelastic, an increase in the number of firms raises social welfare. If in addition the distribution of types is log concave, then an increase in the number of firms raises aggregate consumer surplus and lowers producer surplus. We identify conditions, however, under which uninformed consumers are harmed, and informed consumers are helped, when the number of firms is larger. By contrast, when the number of firms is held fixed, a policy that increases the share of informed consumers will benefit informed and uninformed consumers. Finally, we confirm that results previously obtained in Varian’s (1980) complete-information model can be captured in our model as a limiting case when the support of possible cost types approaches zero.

We close by mentioning a possible extension of the model. The model considered above assumes that consumers are unable to engage in sequential search. We may question how an increase in $N$ affects equilibrium behavior when this assumption is relaxed by allowing that, after an uninformed consumer visits a firm and observes that firm’s price, the consumer may elect to incur a search cost and visit another firm. In fact, Stahl (1989) addresses this question in a complete-information setting. An interesting direction for future work is to consider the impact of $N$ on pricing and welfare in an incomplete-information setting when sequential search by uninformed consumers is allowed. We plan to explore this extension in future research.\footnote{See also Janssen, Pichler and Weidenholzer (2011).}

7 Proofs

Proof of Proposition 1. We now show (i) that in any equilibrium $p(\theta)$ must be differentiable and (ii) that $p(\theta)$ defined by (1) is an equilibrium pricing function. Uniqueness follows since any equilibrium pricing function must satisfy the boundary condition, $p(\bar{\theta}) = p^m(\bar{\theta})$.

(i) Differentiability: For this proof, notice first that $p(\theta)$ is continuous. If $p(\theta)$ were to
have a jump, then a firm at the jump point would deviate without affecting the probability of winning informed consumers. Suppose next \( \hat{\theta} > \theta \). Following Maskin and Riley (1984), we may use the mean-value theorem and write On-IC(\( \theta \)) as

\[
[M(\theta) - M(\hat{\theta})] \pi(p(\theta), \theta) - M(\hat{\theta}) \pi_p(p(\theta^*), \theta) [p(\hat{\theta}) - p(\theta)] \geq 0,
\]

where \( p(\theta^*) \in (p(\theta), p(\hat{\theta})) \) is chosen to satisfy

\[
\pi_p(p(\theta^*), \theta) = \frac{\pi(p(\hat{\theta}), \theta) - \pi(p(\theta), \theta)}{p(\hat{\theta}) - p(\theta)}.
\]

This is possible since \( p(\theta) \) is continuous and \( \pi(\rho, \theta) = (\rho - \theta)D(\rho) \) is differentiable in \( \rho \).

Similarly, we describe On-IC(\( \hat{\theta} \)) as

\[
[M(\hat{\theta}) - M(\theta)] \pi(p(\hat{\theta}), \hat{\theta}) - M(\theta) \pi_p(p(\theta^{**}), \hat{\theta}) [p(\theta) - p(\hat{\theta})] \geq 0,
\]

where \( p(\theta^{**}) \in (p(\theta), p(\hat{\theta})) \) is chosen to satisfy

\[
\pi_p(p(\theta^{**}), \hat{\theta}) = \frac{\pi(p(\theta), \hat{\theta}) - \pi(p(\hat{\theta}), \hat{\theta})}{p(\theta) - p(\hat{\theta})}.
\]

Then, for \( \hat{\theta} > \theta \) and \( \hat{\theta} - \theta \) sufficiently small, On-IC(\( \theta \)) and On-IC(\( \hat{\theta} \)) become

\[
-\frac{M(\hat{\theta})-M(\theta)}{\hat{\theta}-\theta} \pi_p(p(\theta^*), \theta) \geq \frac{p(\hat{\theta}) - p(\theta)}{\hat{\theta} - \theta} \geq -\frac{M(\hat{\theta})-M(\theta)}{\hat{\theta}-\theta} \pi_p(p(\theta^{**}), \hat{\theta}).
\]

Similarly, for \( \hat{\theta} < \theta \) and \( \theta - \hat{\theta} \) sufficiently small, On-IC(\( \theta \)) and On-IC(\( \hat{\theta} \)) become

\[
-\frac{M(\hat{\theta})-M(\theta)}{\theta-\hat{\theta}} \pi_p(p(\theta^*), \theta) \leq \frac{p(\hat{\theta}) - p(\theta)}{\theta - \hat{\theta}} \leq -\frac{M(\hat{\theta})-M(\theta)}{\theta-\hat{\theta}} \pi_p(p(\theta^{**}), \hat{\theta}).
\]

Thus, taking limits as \( \hat{\theta} \to \theta \) from both sides and using the differentiability of \( M(\theta) \), we can find the derivative:

\[
p'(\theta) = -\frac{\pi(p(\theta), \theta) \frac{\partial M(\theta)}{\partial \theta}}{\pi_p(p(\theta^*), \theta) M(\theta)}.
\]

Hence, \( p(\theta) \) is differentiable. Since \( p(\theta) = p^m(\theta) \), we see that \( p(\theta) \) is uniquely defined.

(ii) **Sufficiency:** For this proof, we show that the price function \( p(\theta) \) defined in (1) is
the equilibrium price function. To this end, we need to show that

$$\Pi(\theta, \hat{\theta}; p) \geq \Pi(\hat{\theta}, \theta; p)$$

for all $\theta$ and $\hat{\theta}$.

Suppose $\hat{\theta} > \theta$. Since $\Pi_1(x, x; p) = 0$ under (1), we have

$$\Pi(\theta, \theta; p) - \Pi(\hat{\theta}, \theta; p) = - \int_{\theta}^{\hat{\theta}} \Pi_1(x, \theta; p) dx$$

$$= \int_{\theta}^{\hat{\theta}} [\Pi_1(x, x; p) - \Pi_1(x, \theta; p)] dx$$

$$= \int_{\theta}^{\hat{\theta}} \int_{\theta}^{x} \Pi_{12}(x, y; p) dy dx > 0,$$

where the inequality follows given $\hat{\theta} > \theta$ and $x > \theta$, since

$$\Pi_{12}(x, y; p) = - \frac{\partial}{\partial x} D(p(x)) M(x) > 0.$$

Thus, $\Pi(\theta, \theta; p) > \Pi(\hat{\theta}, \theta; p)$ for $\hat{\theta} > \theta$. Similarly, we can show that $\Pi(\theta, \theta; p) > \Pi(\hat{\theta}, \theta; p)$ for $\hat{\theta} < \theta$. Hence, the price $p(\theta)$ defined in (1) satisfies On-IC($\theta$). As we show in the text, if $p(\theta)$ satisfies On-IC($\theta$), then it also satisfies Off-IC($\theta$). ■

**Proof of Proposition 2.** For the proof, we characterize the function $\varphi_N(\theta) \equiv \frac{\partial \varphi(\theta)}{\partial N}$. We begin by rewriting $\varphi_N(\theta)$ as

$$\varphi_N(\theta) = \frac{U}{N^2} \left( \pi(p^m(\theta), \theta) + \int_{\theta}^{\bar{\theta}} D(p(x)) dx - \pi(p(\theta), \theta) \right)$$

$$+ \pi(p(\theta), \theta) [1 - F(\theta)]^{N-1} I \ln [1 - F(\theta)]$$

$$- \int_{\theta}^{\bar{\theta}} D(p(x))[1 - F(x)]^{N-1} I \ln [1 - F(x)] dx - \int_{\theta}^{\bar{\theta}} \frac{\partial p(x)}{\partial N} M(x) dx \right).$$

(22)

If $x \to \bar{\theta}$, then the integrand in the last term in (22) approaches zero. This follows since, if $x \to \bar{\theta}$, then $p(x) \to p^m(x)$ and so $\frac{\partial p(x)}{\partial N} \to 0$. Since $[1 - F(\theta)]^{N-1} \ln [1 - F(\theta)] \to 0$ as $\theta \to \bar{\theta}$, we can thus show that the endpoint at the top satisfies $\varphi_N(\bar{\theta}) = 0$. To characterize
\[ \varphi_N(\theta) \text{ for } \theta < \bar{\theta}, \text{ we derive that} \]

\[
\frac{\partial \varphi_N(\theta)}{\partial \theta} = -\pi_p(p(\theta), \theta)p'(\theta) \left( \frac{U}{N^2} - [1 - F(\theta)]^{N-1} I \ln [1 - F(\theta)] \right) + \frac{\pi(p(\theta), \theta)}{\pi_p(p(\theta), \theta)} [1 - F(\theta)]^{N-2} f(\theta) I \left[ (N - 1) \ln [1 - F(\theta)] + 1 \right] - \frac{\varphi_N(\theta)D'(p(\theta))}{\pi_p(p(\theta), \theta)}.\]

Using the differential equation in (1), we rewrite the RHS as

\[
\frac{\partial \varphi_N(\theta)}{\partial \theta} = -\pi_p(p(\theta), \theta)p'(\theta)\eta(\theta) - \frac{\varphi_N(\theta)D'(p(\theta))}{\pi_p(p(\theta), \theta)}, \tag{23}
\]

where

\[ \eta(\theta) = \frac{U}{N} \left( \frac{1}{N} + \frac{1}{N-1} \right) + \frac{U}{N} \ln [1 - F(\theta)] + \frac{I}{N-1} [1 - F(\theta)]^{N-1}. \]

The function \( \eta(\theta) \) is strictly decreasing in \( \theta \) and has a unique type \( \theta^{**} \in (\bar{\theta}, \bar{\theta}) \) such that \( \eta(\theta^{**}) = 0 \), \( \eta(\theta) > 0 \) for \( \theta < \theta^{**} \) and \( \eta(\theta) < 0 \) for \( \theta > \theta^{**} \).

We now use (23) to establish two findings. Our first finding is as follows: For all \( \theta \in [\theta^{**}, \bar{\theta}) \), \( \varphi_N(\theta) < 0 \). We prove this finding in two steps. First, assume to the contrary that there exists \( \tilde{\theta} \in (\theta^{**}, \bar{\theta}) \) such that \( \varphi_N(\tilde{\theta}) \geq 0 \). Since \( \eta(\tilde{\theta}) < 0 \), we then have from (23) that \( \frac{\partial \varphi_N(\theta)}{\partial \theta} > 0 \) for all \( \theta \in (\tilde{\theta}, \bar{\theta}) \) and so \( \varphi_N(\theta) > 0 \) for all \( \theta \in (\tilde{\theta}, \bar{\theta}) \). The contradiction is now apparent, since it is then impossible for \( \varphi_N(\theta) \) to approach the endpoint \( \varphi_N(\bar{\theta}) = 0 \) when \( \theta \rightarrow \bar{\theta} \). Second, assume to the contrary that \( \varphi_N(\theta^{**}) \geq 0 \). If \( \varphi_N(\theta^{**}) > 0 \), then it must be that \( \varphi_N(\theta) > 0 \) for some \( \tilde{\theta} \in (\theta^{**}, \bar{\theta}) \), which leads to a contradiction as in the first step. Suppose then that \( \varphi_N(\theta^{**}) = 0 \). Since \( \eta(\theta^{**}) = 0 \), we then see from (23) that \( \frac{\partial \varphi_N(\theta^{**})}{\partial \theta} = 0 \). But using \( \eta(\theta) < 0 \), we may establish that \( \frac{\partial^2 \varphi_N(\theta^{**})}{\partial \theta^2} > 0 \) in this case. Thus, we conclude that \( \varphi_N(\theta) > 0 \) for some \( \tilde{\theta} \in (\theta^{**}, \bar{\theta}) \), which again leads to a contradiction as in the first step. Our first finding is thus established. Our second finding is as follows: If there exists \( \tilde{\theta} \in (\bar{\theta}, \theta^{**}) \) such that \( \varphi_N(\tilde{\theta}) = 0 \), then \( \frac{\partial \varphi_N(\tilde{\theta})}{\partial \theta} < 0 \). Using (23), we see that this finding holds since, if \( \tilde{\theta} \in (\bar{\theta}, \theta^{**}) \) and \( \varphi_N(\tilde{\theta}) = 0 \), then \( \eta(\tilde{\theta}) > 0 \) and so \( \frac{\partial \varphi_N(\tilde{\theta})}{\partial \theta} < 0 \).

We next show that the endpoint at the bottom satisfies \( \varphi_N(\bar{\theta}) > 0 \). Suppose that \( \varphi_N(\bar{\theta}) \leq 0 \). Then the findings presented above imply that \( \varphi_N(\theta) \leq 0 \) for all \( \theta \in (\bar{\theta}, \bar{\theta}) \). We know that the interim-stage profit for \( \bar{\theta} \) is

\[
\pi(p(\bar{\theta}), \bar{\theta}) = \pi(p^m(\bar{\theta}), \bar{\theta}) \frac{U}{M(\bar{\theta})} + \int_{\theta}^{\bar{\theta}} D(p(x)) \frac{M(x)}{M(\bar{\theta})} dx. \tag{24}
\]
The RHS of (24) is strictly decreasing in \( N \): given \( M(\theta) = \frac{U}{N} + I \), \( \frac{U/N}{M(\theta)} \) and \( \frac{M(\theta)}{M(\bar{\theta})} \) are strictly decreasing in \( N \) for \( x > \theta \), and given \( \varphi_N(\theta) \leq 0 \) for all \( \theta \in (\theta, \bar{\theta}) \),

\[
\frac{\partial D(p(x))}{\partial N} = D'(p(x)) \frac{\partial p(x)}{\partial N} = -D'(p(x)) \frac{\varphi_N(x)}{\pi_{p}(p(x), x) M(x)} \leq 0 \quad \text{for all } x \in (\theta, \bar{\theta}).
\]

Since the RHS of (24) is strictly decreasing in \( N \), \( p(\bar{\theta}) \) is strictly decreasing in \( N \), which contradicts the inequality \( \varphi_N(\theta) \leq 0 \). Thus, \( \varphi_N(\theta) > 0 \).

We may now establish the main result: there exists a unique type \( \theta^* \in (\theta, \theta^{**}) \) such that \( \varphi_N(\theta) > 0 \) for \( \theta \in [\theta, \theta^*), \varphi_N(\theta) < 0 \) for \( \theta \in (\theta^*, \bar{\theta}) \) and \( \varphi_N(\theta) = 0 \) for \( \theta \in \{\theta^*, \bar{\theta}\} \).

Using \( \varphi_N(\theta) > 0 \), \( \varphi_N(\bar{\theta}) = 0 \) and our first finding above, we conclude that there exists \( \theta^* \in (\theta, \theta^{**}) \) such that \( \varphi_N(\theta^*) = 0 \) and that \( \varphi_N(\theta) < 0 \) for all \( \theta \in [\theta^*, \bar{\theta}) \). The result now follows since our second finding above ensures that \( \frac{\partial \varphi_N(\theta^*)}{\partial \theta} < 0 \) for any such \( \theta^* \).

**Proof of Proposition 3.** (i) From \( \eta(\theta^{**}) = 0 \), we show that

\[
\frac{\partial \theta^{**}}{\partial N} = -\frac{\partial \eta(\theta^{**})}{\partial \theta} < 0.
\]

The denominator of (25) is negative, since \( \eta(\theta) \) is strictly decreasing in \( \theta \). We next establish the inequality, \( \frac{\partial \eta(\theta^{**})}{\partial N} < 0 \), where

\[
\frac{\partial \eta(\theta^{**})}{\partial N} = -\frac{U}{N^2} \left( \frac{1}{N} + \frac{1}{N-1} \right) - \frac{U}{N} \left( \frac{1}{N^2} + \frac{1}{(N-1)^2} \right) - \frac{U}{N^2} \ln[1 - F(\theta^{**})]
\]

\[
- \frac{(1 - F(\theta^{**}))^{N-1} I}{(N-1)^2} + \frac{(1 - F(\theta^{**}))^{N-1} I \ln[1 - F(\theta^{**})]}{N-1}.
\]

From \( \eta(\theta^{**}) = 0 \), we derive a term:

\[
\frac{U}{N} \ln[1 - F(\theta^{**})] = -\frac{U}{N} \left( \frac{1}{N} + \frac{1}{N-1} \right) - \frac{(1 - F(\theta^{**}))^{N-1} I}{N-1}.
\]

Plugging this term into (26), we confirm that \( \frac{\partial \eta(\theta^{**})}{\partial N} < 0 \). To confirm that \( \theta^{**} \to \theta \) and thus \( \theta^* \to \theta \) as \( N \to \infty \), we observe further that \( \eta(\theta) \to 0 \) and \( \eta'(\theta) \to -f(\theta)I < 0 \) as \( N \to \infty \). \( \blacksquare \)

**Proof of Proposition 6.** Part (i) follows from arguments in the text. To prove part (ii), we first observe that \( \int_{\theta}^{\bar{\theta}} \frac{U}{M(\theta)} dF(\theta) = \int_{0}^{1} \frac{1}{1 + \frac{I}{U} N x^{N-1}} dx \) for any \( F \), and show that

\[
\frac{\partial}{\partial N} \int_{0}^{1} \frac{1}{1 + \frac{I}{U} N x^{N-1}} dx > 0.
\]
To establish the inequality in (27), we utilize the complete-information analysis by Morgan, Orzen and Sefton (2006). When demand is inelastic, they find for the symmetric mixed-strategy equilibrium \( \Phi(p) \) defined in (29) below that \( \frac{\partial P^U}{\partial N} > 0 \) where \( P^U_c = \int_{p}^{r} p d\Phi(p) \).

28 While they consider an inelastic demand setting in which the reservation value is unity, \( r = 1 \), and production is costless, \( c = 0 \), it is straightforward to confirm that this finding also holds when \( r > c > 0 \).

We can then claim that the inequality \( \frac{\partial P^U}{\partial N} > 0 \) established by Morgan, Orzen and Sefton (2006), is equivalent to the inequality (27). This claim is immediately ensured by

\[
\int_{p}^{r} p d\Phi(p) = c + \pi(r, c) \int_{p}^{r} \frac{U/N}{U/N + [1 - \Phi(p)]^{N-1}} d\Phi(p)
\]

\[
= c + \pi(r, c) \int_{0}^{1} \frac{1}{1 + \frac{1}{N} x^{N-1}} dx,
\]

where the second equality follows after a change of variables.

We now know that \( \int_{\theta}^{\bar{\theta}} \frac{U/N}{M(\theta)} dF(\theta) \) is independent of \( \bar{\theta} - \theta \) and

\[
\frac{\partial}{\partial N} \int_{\theta}^{\bar{\theta}} \frac{U/N}{M(\theta)} dF(\theta) = \frac{\partial}{\partial N} \int_{0}^{1} \frac{1}{1 + \frac{1}{N} x^{N-1}} dx > 0.
\]

We next complete the proof by showing that for any \( N \), if \( \bar{\theta} - \theta \) approaches zero, then so does the term:

\[
\frac{\partial}{\partial N} \int_{\theta}^{\bar{\theta}} \int_{0}^{\bar{\theta}} \frac{M(x)}{M(\theta)} dx dF(\theta) = \int_{\theta}^{\bar{\theta}} \int_{0}^{\bar{\theta}} \rho(\theta, x) dx dF(\theta),
\]

where

\[
\rho(\theta, x) = \frac{\partial}{\partial N} \frac{M(x)}{M(\theta)}.
\]

We first find that the integrand,

\[
\rho(\theta, x) = \frac{\frac{\partial}{\partial N} M(x) M(\theta) - \frac{\partial}{\partial N} M(\theta) M(x)}{[M(\theta)]^2},
\]

is finite for all \( \theta \in [\theta, \bar{\theta}] \) and \( x \in [\theta, \bar{\theta}] \). The denominator is bounded by the range \([((\frac{U}{N})^2, 1]\). The numerator has all finite terms; the term,

\[
\frac{\partial M(x)}{\partial N} = - \frac{U}{N^2} + [1 - F(x)]^{N-1} I \ln[1 - F(x)],
\]

has the maximum \(- \frac{U}{N^2}\) at \( x \in \{\theta, \bar{\theta}\} \) and the minimum \(- \frac{U}{N^2} - \frac{1}{(N-1)} e^{-1}\) at \( x = F^{-1}(1 -
Thus, from the space of \((\theta, x)\), we can select a pair \((\theta^*, x^*)\) that maximizes the absolute value of the integrand. Note also that, for each term in the numerator or denominator of \(\rho(\theta, x)\), the support of possible values for that term is independent of the support of cost types, \([\bar{\theta}, \bar{\theta}]\); hence, we may bound the maximum of the absolute value of \(\rho(\theta, x)\) independently of the support of cost types. Now, given \(|\rho(\theta^*, x^*)| < \infty\), we have

\[
\int_{\bar{\theta}}^{\bar{\theta}} \int_{\bar{\theta}}^{\bar{\theta}} \frac{\partial}{\partial N} \frac{M(x)}{M(\theta)} d\theta dF(\theta) < \int_{\bar{\theta}}^{\bar{\theta}} \int_{\bar{\theta}}^{\bar{\theta}} |\rho(\theta^*, x^*)| d\theta dF(\theta) < |\rho(\theta^*, x^*)|(\bar{\theta} - \bar{\theta}),
\]

and conclude that for any \(N\), if \(\bar{\theta} - \bar{\theta}\) approaches zero, then \(\frac{\partial}{\partial N} \int_{\bar{\theta}}^{\bar{\theta}} \frac{M(x)}{M(\theta)} d\theta dF(\theta)\) approaches zero.

We conclude the proof by establishing part (iii). Our strategy is to show that, for given \(F\) and \(N\), \(\frac{\partial P}{\partial N}\) approaches zero as \(\bar{\theta} - \bar{\theta}\) goes to zero. Once this is shown, we can conclude that part (iii) follows from part (ii). As argued in the text, we know that \(P = rU + I \int_{\bar{\theta}}^{\bar{\theta}} \theta dG(\theta) + I \int_{\bar{\theta}}^{\bar{\theta}} \int_{\bar{\theta}}^{\bar{\theta}} N(1 - F(x))^{N-1} d\theta dF(\theta)\).

As in the text, we can rewrite the second and third term on the RHS as

\[
I \int_{\bar{\theta}}^{\bar{\theta}} \theta dG(\theta) = I[\bar{\theta} - \int_{\bar{\theta}}^{\bar{\theta}} G(\theta)d\theta]
\]

and

\[
I \int_{\bar{\theta}}^{\bar{\theta}} \int_{\bar{\theta}}^{\bar{\theta}} N(1 - F(x))^{N-1} d\theta dF(\theta) = IN \int_{\bar{\theta}}^{\bar{\theta}} F(\theta)(1 - F(\theta))^{N-1} d\theta.
\]

We next differentiate \(P\) with respect to \(N\):

\[
\frac{\partial P}{\partial N} = I \int_{\bar{\theta}}^{\bar{\theta}} [1 - F(\theta)]^N \ln[1 - F(\theta)] d\theta + I \int_{\bar{\theta}}^{\bar{\theta}} F(\theta)[1 - F(\theta)]^{N-1} d\theta
\]

\[
+ IN \int_{\bar{\theta}}^{\bar{\theta}} F(\theta)[1 - F(\theta)]^{N-1} \ln[1 - F(\theta)] d\theta
\]

\[
= I \int_{\bar{\theta}}^{\bar{\theta}} [1 - F(\theta) + NF(\theta)][1 - F(\theta)]^{N-1} \ln[1 - F(\theta)] d\theta + I \int_{\bar{\theta}}^{\bar{\theta}} F(\theta)[1 - F(\theta)]^{N-1} d\theta.
\]

Note that each integrand on the RHS is finite; hence, for any \(F\) and \(N\), if \(\bar{\theta} - \bar{\theta}\) approaches zero, then \(\frac{\partial P}{\partial N}\) approaches zero. We now conclude that, for any \(F\) and \(N\), if \(\bar{\theta} - \bar{\theta}\) is sufficiently small, then \(\frac{\partial p_i}{\partial N} > 0\) from part (ii) implies that \(\frac{\partial p_i}{\partial N} < 0\).
**Purification.** We present here a purification result for the symmetric mixed-strategy equilibrium of the Varian-type complete-information game. Bagwell and Wolinsky (2002) present a purification result for the case in which demand is inelastic, and we offer here a slight extension of their analysis to consider the case of downward-sloping demand functions.

Assume that \( N \) firms have the common production cost \( c > 0 \). A pure strategy for firm \( i \) is \( p_i \in [c, P_R] \), where for simplicity \( P_R \) is given by \( D(P_R) = 0 \). Let \( p_{-i} \) denotes \((N - 1)\)-tuple of prices selected by other firms. Firm \( i \)'s profit is then

\[
\Pi_i(p_i, p_{-i}) = \begin{cases} 
\pi(p_i, c) \frac{U}{N} & \text{if } p_i > \min_{j \neq i} p_j \\
\pi(p_i, c) \left( \frac{U}{N} + \frac{1}{k} \right) & \text{if } p_i \leq \min_{j \neq i} p_j \text{ and } \{ j : p_j = p_i \} = k - 1.
\end{cases}
\]

Assume that \( \pi(p_i, c) \equiv (p_i - c)D(p_i) \) is strictly concave with maximizer \( p^m(c) > c \). For firm \( i \), a mixed strategy is a distribution function \( \Phi_i \) defined over the support \([p, \overline{p}]\), where \( \overline{p} \) and \( p \) are defined below. Firm \( i \)'s profit is then

\[
E_i(\Phi_i, \Phi_{-i}) = \int_{\overline{p}} \cdots \int_{\overline{p}} \Pi(p_i, p_{-i}) d\Phi_1 \cdots d\Phi_N.
\]

As in Varian (1980) and Rosenthal (1980), we can establish that there is no pure-strategy Nash equilibrium, and that there is a unique symmetric Nash equilibrium, \( \Phi = \Phi_i \) for all \( i \):

\[
\Phi(p) = 1 - \left( \frac{\pi(p^m(c), c) - \pi(p, c)}{\pi(p, c)} \frac{U}{IN} \right)^{\frac{1}{k-1}} \forall p \in [p, \overline{p}]
\]

(28)

where \( \overline{p} \) satisfies \( \pi(p, c) = \pi(p^m(c), c) \frac{U/N}{U/N+1} \) and \( \overline{p} = p^m(c) \). The equilibrium is established by the iso-profit condition for \( p \in [p, \overline{p}] \):

\[
\pi(p, c) \left( \frac{U}{N} + [1 - \Phi(p)]^{N-1}I \right) = \pi(p^m(c), c) \frac{U}{N}.
\]

(29)

We next consider an incomplete-information game, where unit costs are constant in output but rise in types \( \theta \). We follow closely related arguments by Bagwell and Lee (2010) and Bagwell and Wolinsky (2002), and show that, if each firm with type \( \theta \) chooses \( p(\theta) \), which is the unique equilibrium in the incomplete-information game, then the probability distribution induced by \( p \) is approximately the distribution of prices in the mixed-strategy equilibrium of the complete-information game with constant cost \( c \), when the payoff relevance of types \( \theta \) gets small. Assume that the unit cost function \( c(\theta) \) is differentiable and strictly increasing in \( \theta \) with \( 0 < c(\theta) < c(\overline{\theta}) < P_R \). Then, as in the text, there exists a
unique equilibrium $p$ that satisfies

$$p'(\theta) = \frac{\pi(p(\theta), \theta)}{\pi_p(p(\theta), \theta)} \left( \frac{U}{N} + [1 - F(\theta)]^{N-1} \right)$$

and $p(\theta) = p^m(\theta)$, \hspace{1cm} (30)

where $\pi(p(\theta), \theta) = [p(c(\theta)) - c(\theta)] D(p(c(\theta)))$.

**Lemma A1.** Given a constant $c \in (0, P_R)$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|c(\theta) - c| < \delta$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, then the probability distribution induced by the equilibrium $p(\theta)$ is $\varepsilon$-close to the mixed-strategy equilibrium of the complete-information game with the common cost type $c$.

**Proof.** Since $p(\theta)$ is strictly increasing, the distribution induced by $p(\theta)$ is $\text{prob} (\theta \mid p(\theta) \leq x) = F(p^{-1}(x))$. Let $\Phi_c$ denote the symmetric mixed-strategy equilibrium of the complete-information game with the common cost $c$. Define the function $p_c$ by $p_c(\theta) \equiv \Phi_c^{-1}(F(\theta))$. The proof is established as an implication of the following three results. First, if each firm with type $\theta$ chooses $p_c(\theta)$, then the distribution of prices is $\Phi_c$. Put differently, $p_c(\theta)$ induces the same distribution of prices as does $\Phi_c$:

\[
\text{prob} (\theta \mid p_c(\theta) \leq x) = \text{prob} (\theta \mid \Phi_c^{-1}(F(\theta)) \leq x) = \text{prob} (\theta \mid \theta \leq F^{-1}(\Phi_c(x))) = F(F^{-1}(\Phi_c(x))) = \Phi_c(x).
\]

Second, if $c(\theta) = c$, then $p_c(\theta)$ solves (30). To show this, we observe that the definition of $p_c(\theta)$ gives that

$$p'_c(\theta) = \frac{f(\theta)}{\Phi'_c(p_c(\theta))}.$$

To find $\Phi'_c(p_c(\theta))$, we refer to the mixed strategy in (29), which holds as an identity for all $p \in [\underline{p}, \overline{p}]$. We may thus differentiate (29) with respect to $p$, which yields that

$$\pi_p(p, c) \left( \frac{U}{N} + [1 - \Phi(p)]^{N-1} \right) = \pi(p, c)(N - 1) [1 - \Phi(p)]^{N-2} \Phi'(p) I.$$

Substituting $\Phi = \Phi_c$ and $p = p_c(\theta)$, we obtain

$$p'_c(\theta) = \frac{\pi(p_c(\theta), c)(N - 1) [1 - F(\theta)]^{N-2} f(\theta) I}{\pi_p(p_c(\theta), c) \left( \frac{U}{N} + [1 - F(\theta)]^{N-1} \right)}.$$

Note that $p_c(\theta) = \Phi_c^{-1}(F(\theta)) = \Phi_c^{-1}(1)$ and thus $\Phi_c(p_c(\theta)) = 1$, which means $p_c(\theta) = \Phi_c^{-1}(F(\theta)) = \Phi_c^{-1}(1)$.
Thus, we may now conclude that, if \( c(\theta) = c \), then \( p_c(\theta) \) solves (30). Third, we claim that, if \(|c(\theta) - c|\) is small, then \( p(\theta) \) induces approximately the same distribution of prices as does \( \Phi_c \). This result builds on the previous results. Our first result tells us that \( p_c(\theta) \) induces \( \Phi_c \). Our second result indicates that \( p_c(\theta) \) approximates \( p(\theta) \) when \( c(\theta) \) approaches \( c \); formally, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \(|c(\theta) - c| < \delta \) for all \( \theta \in [\bar{\theta}, \tilde{\theta}] \), then \(|p(\theta) - p_c(\theta)| < \varepsilon \). In short, as \( c(\theta) \) approximates \( c \), the type \( \theta \) becomes less payoff-relevant, and so the distribution induced by \( p(\theta) \), \( \text{prob}(\theta \mid p(\theta) \leq x) \), approximates \( \Phi_c \).

8  Table

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9  References


