Generalized Delegation

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Abstract

We provide a general sufficient theorem to solve the delegation problem. For interval allocations, our theorem's conditions align with previous literature. Significantly, our theorem extends beyond this, successfully addressing cases with discontinuous allocations. We demonstrate the theorem's power through examples, connecting with existing results and uncovering novel optimal allocation structures.

1 Introduction

In the standard delegation problem, a principal must decide how much discretion to grant to an agent who chooses an action. The trade-off confronting the principal is that the agent has superior information about the state of the world but is biased (i.e., does not share the principal's preferences). A defining feature of the delegation problem is that the principal is unable to use contingent transfers to shape the agent's incentives; thus, in the standard formulation, the principal selects a set of permissible actions from the real line, and the agent picks an action from this set. For example, the principal may utilize "interval delegation" whereby the set of permissible actions is an interval. Alternatively, the set of permissible actions may contain only a finite number of distinct actions. Hybrid sets that include distinct actions and intervals are also possible.

The delegation model begins with Holmström (1977) and is now a workhorse model in economics. The applications of the delegation framework are wide-ranging and include consumption-savings (Amador, Werning, and Angeletos, 2006), regulation (Alonso and Matouschek, 2008; Amador and Bagwell, 2022; Kolotilin and Zapechelnyuk, 2024), veto bargaining (Mylovanov, 2008; Kartik, Kleiner, and Van Weelden, 2021), tariff caps (Amador and Bagwell, 2013), fiscal

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rules (Halac and Yared, 2014, 2022; Sublet, 2023), and quality certification (Zapechelnyuk, 2020), for example. A central goal for the delegation literature has been to determine general sufficient conditions under which interval delegation is optimal among all incentive-compatible allocations. Melumad and Shibano (1991), Alonso and Matouschek (2008) and Amador and Bagwell (2013) provide findings of this kind.

Amador and Bagwell (2013) analyze the optimality of interval delegation in the context of a general representation of the delegation problem.[1] The agent's action is taken from an interval on the real line, the state has a continuous distribution over a bounded interval on the real line, the principal's payoff function is continuous in the action and state and twice differentiable and concave in the action, and the agent's welfare function is twice differentiable and concave in the action with the state entering this function in a standard multiplicative fashion. Using and extending the Lagrangian methods developed by Amador, Werning, and Angeletos (2006), Amador and Bagwell provide sufficient conditions for optimal delegation to take the form of interval delegation. They also identify a preference family under which their sufficient conditions are necessary for the optimality of interval delegation.

Although the applications for interval delegation are extensive, several important applications feature other forms of delegation. As Saran (2024) discusses, volunteer organizations, such as hospitals, can require volunteers to choose from predetermined shifts, and charities can have donation campaigns that feature a finite menu of suggested donation amounts. The resulting allocations are discontinuous with respect to the agent's information or type. Sufficient conditions for the optimality of discontinuous allocations are less well understood, although the early work by Melumad and Shibano (1991) provides results for a specific setting (quadratic preferences, uniform distribution). As discussed in the following, Saran's recent work also provides sufficient conditions.

In this paper, we extend the Lagrangian approach and provide general sufficient conditions for optimal delegation. We capture Amador and Bagwell (2013) sufficient conditions for the optimality of interval delegation as a special case, but crucially, our analysis is not limited to settings in which interval delegation is optimal. We also provide sufficient conditions for the optimality of incentive-compatible allocations that are discontinuous. Thus, for example, we provide sufficient conditions for the optimality of incentive-compatible allocations that are induced when the permissible set is comprised of a finite number of actions. In addition, we provide sufficient conditions for the optimality of incentive-compatible hybrid allocations that are generated when the permissible set of actions features a combination of distinct actions and intervals.

[1] Amador and Bagwell report their results both when money burning is infeasible, as in the standard delegation problem, and when money burning is allowed. For conditions under which optimal delegation entails actual money burning, see Ambrus and Egorov (2017) and Amador and Bagwell (2020).

To develop these findings, we begin by considering "escalators-and-stairs" allocations. An escalators-and-stairs allocation is defined by a finite partition of the type space into interval subsets such that within any interval, either the same action is allocated to each type or the agent's preferred or "flexible" action is allocated to each type. The former case corresponds to a pooling interval, while the latter case is a separating interval, since in our model the agent's flexible allocation is strictly increasing with respect to the type. Restating a result by Melumad and Shibano (1991), we show that an escalators-and-stairs allocation is incentive compatible if and only if it satisfies the properties that (i) adjacent pooling intervals entail an upward jump at the indifferent type, and (ii) adjacent intervals such that one separates and one pools are continuous with the pooling action thus corresponding to the flexible action for the boundary type.

We next characterize the necessary conditions for an escalators-and-stairs allocation to be optimal within a restricted class of incentive-compatible escalator-and-stairs allocations. The restricted class shares a common partition with the same intervals of separation and pooling. We associate these restrictions with a finite set of scalar multipliers and characterize the first-order conditions using these multipliers.

Armed with these results, we then proceed to our final step and characterize in Theorem 1 sufficient conditions under which a given incentive-compatible escalators-and-stairs allocation is optimal among the class of all incentive-compatible allocations. To do so, and following Amador, Werning, and Angeletos (2006) and Amador and Bagwell (2013), we construct (cumulative) multiplier functions where the independent variable is the type. As we show, these functions can be defined in terms of scalar multipliers for the restricted problem just described. To put Theorem 1 to use, one simply presents a candidate solution that is an incentive-compatible escalators-and-stairs allocation. For the associated pooling and separation intervals, the sufficient conditions then indicate the properties to be satisfied by the welfare functions of the principal and agent and also the distribution function. Importantly, our sufficient conditions can thus be used to address the optimality of interval allocations, pooling allocations that entail a finite number of distinct actions, and hybrid allocations that include both intervals of pooling and separation.

We also use Theorem 1 to characterize sufficient conditions for the optimality of various specific allocations, including those featured in the previous literature. We show that Theorem 1 provides Amador and Bagwell (2013) sufficient conditions for the optimality of interval delegation as a special case. An interval allocation is an escalators-and-stairs allocation with an interval partition of at most three regions, where in the case of three regions, the middle region entails separation, and the exterior regions involve pooling (at the floor and cap, respectively). We also consider the optimality of an extreme one-step allocation, which in our context is an escalators-and-stairs allocation that allows for exactly two actions and thus induces two pooling regions with an upward jump at the indifferent type. Melumad and Shibano (1991) provide sufficient conditions for the optimality of this allocation in linear-quadratic pay-offs and uniformly distributed types. We show that Theorem 1 also provides these sufficient conditions as a special case. As a further illustration of the reach of Theorem 1, we consider a specific hybrid example in which the optimal allocation has both separating and adjacent pooling regions. Finally, we consider an extension of the hybrid example to preferences that fall outside the family of preferences identified by Amador and Bagwell (2013) and emphasized in subsequent work, thus confirming that the applicability of Theorem 1 extends beyond this preference family.

Three recent contributions are important to mention. Kolotilin and Zapechelnyuk (2024) present a clever argument that maps the delegation problem to an optimal persuasion problem. Kleiner, Moldovanu, and Strack (2021) show how to use powerful tools from majorization to solve the delegation problem. These two papers have significantly expanded the set of tools available to tackle delegation problems. Our paper complements and provides an alternative approach. Relying on our Lagrangian approach, we uncover similar results as these papers do, but as noted in certain cases, we can go beyond the family of preferences that both of these papers rely on.[2] Finally, Saran (2024) considers a general delegation problem with a participation constraint and establishes conditions under which the optimal allocation has a finite number of jumps. He also numerically solves for the optimal allocation under specific utility functions. Our approach differs in that we use a Lagrangian approach to provide general analytical characterizations of optimal allocations for a setting without a participation constraint.

As a further extension of our analysis, we extend our sufficient condition to the case where money burning is feasible. In this way, we establish the optimality of escalator-and-stairs allocations even when money burning is feasible. The extension of our results follows the approach used by Amador and Bagwell (2013) in establishing conditions for the optimality of interval allocations when money burning is feasible.

The remainder of the paper is structured as follows. Section 2 presents the model. Sections 3 and 4 present the types of allocations that are incentive compatible (escalators-and-stairs) and discuss the necessary conditions. Section 5 presents the sufficient conditions and our main result, Theorem 1. Section 6 discusses the results and relates our findings to the literature. Section 7 extends the analysis to allow money burning. Section 8 concludes.

[2]Kleiner (2022) is also relevant. He extends the delegation literature to multidimensional settings under the assumption that stochastic mechanisms are available. Our approach is again complementary and utilizes an alternative Lagrangian approach. For the single-dimensional setting, as noted, we also report results that go beyond the family of preferences assumed by Kleiner.

2 Model

We consider a standard delegation problem with a principal and an agent. The agent observes the state or type $\gamma \in \Gamma \equiv [\underline{\gamma}, \overline{\gamma}]$ with $\overline{\gamma} > \underline{\gamma}$ drawn from a continuous distribution F with an associated continuous density f. The agent chooses an action π. The action is restricted to be within a feasible set $\Pi = [0, \bar{\pi}]$ with $\bar{\pi} > 0$. The payoff to an agent of type γ that undertakes action π is given by $\gamma\pi + b(\pi)$. Given a realization of type γ and an action π, the principal receives an interim payoff $w(\gamma, \pi)$. For the remainder of the paper, we impose the following conditions.

Assumption 1. *The following holds: (i) the function $w : \Gamma \times \Pi \to \mathbb{R}$ is continuous on $\Gamma \times \Pi$; (ii) for any $\gamma_0 \in \Gamma$, the function $w(\gamma_0, \cdot)$ is concave on Π, and twice differentiable on $(0, \bar{\pi})$; (iii) the function $b : \Pi \to \mathbb{R}$ is strictly concave on Π, and twice differentiable on $(0, \bar{\pi})$; (iv) there exists a twice differentiable function $\pi^f : \Gamma \to (0, \bar{\pi})$ such that, for all $\gamma_0 \in \Gamma$, $\pi^f(\gamma_0)$ is strictly increasing and $\pi^f(\gamma_0) \in \arg\max_{\pi\in\Pi} \{\gamma_0\pi + b(\pi)\}$; and (v) the function $w_\pi : \Gamma \times (0, \bar{\pi}) \to \mathbb{R}$ is continuous on $\Gamma \times (0, \bar{\pi})$, where w_π denotes the derivative of w in its second argument.*

Note that $\pi^f(\gamma)$ denotes the agent's preferred action given the private information γ. Assumption (iv) restricts attention to preferences such that π_f is strictly increasing in γ.

In the delegation problem, the principal does not observe the realization of the type γ. However, the principal can restrict the choice set of the agent. As is standard, we write the principal problem as a mechanism design problem. The principal chooses an allocation $\boldsymbol{\pi}$ which is a function $\boldsymbol{\pi} : \Gamma \to \Pi$ that determines the action given the private information of the agent. The goal is to choose $\boldsymbol{\pi}$ so as to maximize the principal's welfare function subject to the incentive constraints of the agent:

$$\max_{\{\boldsymbol{\pi}:\Gamma\to\Pi\}} \int_\Gamma w(\gamma, \boldsymbol{\pi}(\gamma)) f(\gamma) d\gamma \quad \text{subject to:} \tag{1}$$
$$\gamma \in \arg\max_{\tilde{\gamma}\in\Gamma} \{\gamma\boldsymbol{\pi}(\tilde{\gamma}) + b(\boldsymbol{\pi}(\tilde{\gamma}))\}$$

where the constraint is an incentive-compatibility constraint that arises since the agent is privately informed of the value of γ. It follows from the supermodularity of the agent's payoff that an incentive-compatible allocation $\boldsymbol{\pi}(\gamma)$ must be increasing in γ.[3]

A particularly simple allocation is the "interval allocation." Amador and Bagwell (2013) obtain general sufficient conditions for interval delegation to be optimal. This corresponds to a case where the optimal allocation $\boldsymbol{\pi}^\star$ is defined by $\gamma_L < \gamma_H$ such that $\boldsymbol{\pi}^\star(\gamma) = \pi^f(\gamma)$ for $\gamma \in [\gamma_L, \gamma_H] \subset$

[3] See Milgrom and Segal (2002).

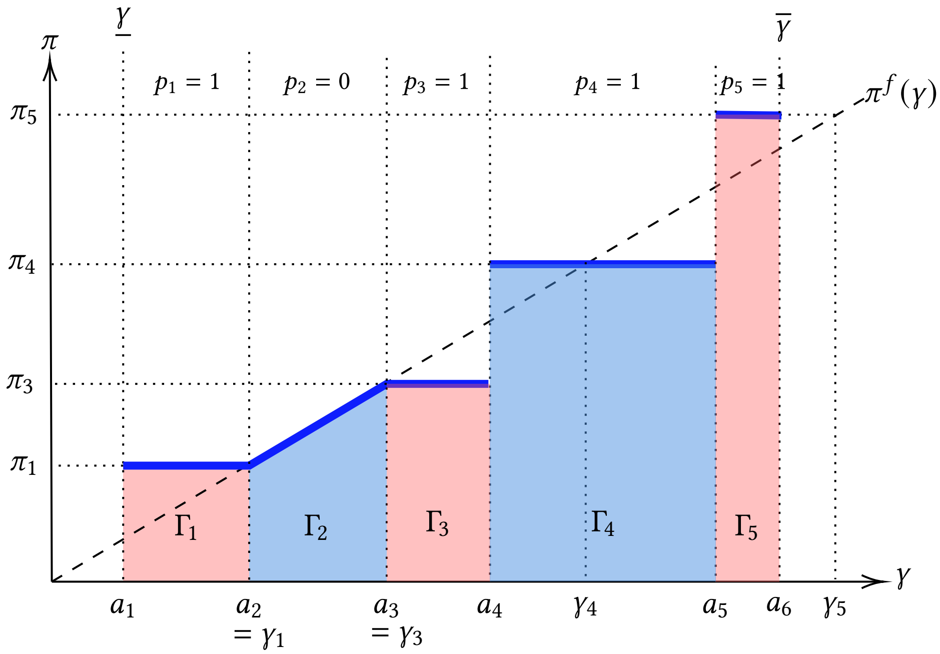

Figure 1: An escalators-and-stairs allocation.
Note: Some of the notation showcased in this figure is explained in Section 4.

Γ and $\boldsymbol{\pi}^\star(\gamma) = \pi^f(\gamma_L)$ for $\gamma < \gamma_L$ and $\boldsymbol{\pi}^\star(\gamma) = \pi^f(\gamma_H)$ for $\gamma > \gamma_H$. Our goal in this paper is to obtain sufficient conditions for other allocations to be optimal, including those that are not of the interval form.

3 Escalators and stairs

Consider an interval $\tilde{\Gamma} \subset \Gamma$. We say that an allocation $\boldsymbol{\pi}$ *separates* in $\tilde{\Gamma}$ if $\boldsymbol{\pi}(\gamma) = \pi^f(\gamma)$ for all $\gamma \in \tilde{\Gamma}$. We say that an allocation *pools at* π in $\tilde{\Gamma}$ if $\boldsymbol{\pi}(\gamma) = \pi$ for all $\gamma \in \tilde{\Gamma}$.

An allocation $\boldsymbol{\pi}$ lies within the "escalators-and-stairs" class if there exists a finite partition of Γ, $\{\Gamma_i\}_{i=1}^n$, with $\Gamma_i \equiv [a_i, a_{i+1}) \subset \Gamma$ for $i \in \{1, \ldots, n-1\}$ and $\Gamma_n \equiv [a_n, a_{n+1}] \subset \Gamma$ and $a_i < a_{i+1}$, such that $\cup_{i \in \{1,\ldots,n\}} \Gamma_i = \Gamma$ and for every $i \in \{1, \ldots, n\}$, $\boldsymbol{\pi}$ either *separates* or *pools* in Γ_i. It is without loss of generality to consider allocations within this class such that no two adjacent intervals are separating, and no two adjacent intervals are pooling at the same action. We maintain these requirements in what follows.

Incentive compatibility imposes additional restrictions on an "escalators-and-stairs" allocation. In particular, consider the following condition:

Condition 1. *An escalators-and-stairs allocation* $\boldsymbol{\pi}$ *with the associated interval partition* $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ *satisfies:*

(i) For any two adjacent intervals Γ_i, Γ_{i+1} *that are pooling at* π_i, π_{i+1} *respectively,* $\pi_i < \pi_{i+1}$ *and* $\gamma\pi_i + b(\pi_i) = \gamma\pi_{i+1} + b(\pi_{i+1})$ *for* $\gamma = a_{i+1}$.

(ii) For any two adjacent intervals Γ_i, Γ_{i+1} *such that one of them separates and the other one pools at* π*, it must be that* $\pi = \pi^f(\gamma)$ *where* $\gamma = a_{i+1}$.

Note that part (i) of Condition 1 necessarily requires that for two adjacent intervals that pool at π_i and π_{i+1}, respectively, it must be $\pi_i < \pi^f(\gamma) < \pi_{i+1}$ for $\gamma = a_{i+1}$.

Figure 1 illustrates an escalators-and-stairs allocation. For the types in Γ_1, the allocation requires pooling at the action π_1. Separation is then specified for the types in Γ_2 with each type thus selecting its flexible action, which, as the figure suggests, can be envisioned as moving up an "escalator." The allocation then requires pooling at the action π_3 for types in Γ_3. As required by part (ii) of Condition 1, notice that the allocation is continuous at the lower and upper boundaries of the separating region Γ_2. In contrast, and as required by part (i) of Condition 1, a jump in the allocation occurs at type a_4, with this type being indifferent between pooling at π_3 and π_4. As illustrated in the figure, a jump between two pooling regions gives the appearance of moving up "stairs." For the types in Γ_4, observe that the pooling action π_4 is above (below) the flexible action for lower (higher) types. Finally, another upward jump in the illustrated allocation occurs at type a_5, with type a_5 being indifferent between pooling at π_4 and π_5.

A first result, a restatement of Melumad and Shibano (1991) in our notation, states that Condition 1 is necessary and sufficient for incentive compatibility:

Proposition 1. *An escalators-and-stairs allocation* $\boldsymbol{\pi}$ *is incentive compatible if and only if it satisfies Condition 1.*

Proof. From Milgrom and Segal (2002), we have that incentive compatibility is equivalent to monotonicity of the allocation and the following integral condition:

$$\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)) = \int_{\gamma_0}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})d\tilde{\gamma} + \gamma_0\boldsymbol{\pi}(\gamma_0) + b(\boldsymbol{\pi}(\gamma_0))$$

for all $\gamma \in \Gamma$ and any baseline $\gamma_0 \in \Gamma$.

Consider first the necessity of conditions (i) and (ii) in Condition 1.

Necessity of condition (i). Consider $\gamma_o \in \Gamma_i$ and $\gamma' \in \Gamma_{i+1}$. Then incentive compatibility requires that

$\pi_{i+1} \geq \pi_i$. Given that we impose that no two adjacent intervals pool at the same action, it follows that $\pi_{i+1} > \pi_i$. The integral form of the incentive constraint then requires

$$\gamma' \pi_{i+1} + b(\pi_{i+1}) = \int_{\gamma_o}^{\gamma} \pi_i d\tilde{\gamma} + \int_{\gamma}^{\gamma'} \pi_{i+1} d\tilde{\gamma} + \gamma_o \pi_i + b(\pi_i),$$

for $\gamma = \min\{\Gamma_{i+1}\}$; which delivers

$$\gamma \pi_{i+1} + b(\pi_{i+1}) = \gamma \pi_i + b(\pi_i).$$

Necessity of condition (ii). Without loss of generality assume that Γ_i separates, and Γ_{i+1} pools at x. Let $\gamma_o \in \Gamma_i$ and $\gamma' \in \Gamma_{i+1}$. Then

$$\gamma' x + b(x) = \int_{\gamma_o}^{\gamma} \pi^f(\tilde{\gamma}) d\tilde{\gamma} + \int_{\gamma}^{\gamma'} x d\tilde{\gamma} + \gamma_o \pi^f(\gamma_o) + b(\pi^f(\gamma_o)),$$

for $\gamma = \min\{\Gamma_{i+1}\}$. Taking the limit of the above as $\gamma_o \to \gamma$, and using the continuity of the flexible allocation function π^f, we have

$$\gamma' x + b(x) = \int_{\gamma}^{\gamma'} x d\tilde{\gamma} + \gamma \pi^f(\gamma) + b(\pi^f(\gamma)),$$

which implies $\gamma x + b(x) = \gamma \pi^f(\gamma) + b(\pi^f(\gamma))$. Uniqueness of the flexible allocation implies that $x = \pi^f(\gamma)$.

Sufficiency. Note that an allocation that satisfies conditions (i) and (ii) is monotone. Given that in the areas where the allocation separates, the flexible allocation is offered to the corresponding types, the only concern for incentive compatibility arises for types in pooling regions.
Consider a type γ in a region Γ_i that pools at x^o. Let $\tilde{\gamma} = \sup \Gamma_i$. Define x' to be either $x' = x^o = \pi^f(\tilde{\gamma})$ if Γ_{i+1} separates or $x' \neq x^o$ and given by:

$$\tilde{\gamma} x' + b(x') = \tilde{\gamma} x^o + b(x^o),$$

if Γ_{i+1} pools at x'. Note that in this latter case, $x' > \pi^f(\tilde{\gamma}) > x^o$.
Type $\tilde{\gamma}$ then either prefers the choice of x^o to any alternative (because it is its flexible choice) or is indifferent between x^o and a higher alternative x'. Given that for any type $\gamma' \geq \tilde{\gamma}$ we have that $\boldsymbol{\pi}(\gamma') \geq x'$ (by monotonicity), it follows that type γ must prefer x^o to *any* other action prescribed to any higher type. A similar argument shows that γ also prefers its prescribe action to that of any lower type. □

The next lemma shows that two incentive-compatible escalators-and-stairs allocations that separate and pool in the same subsets and assign the same action to the lowest type are the same:

Lemma 1. *Let $\boldsymbol{\pi}$ be an incentive-compatible escalators-and-stairs allocation with associated partition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$. Let $\hat{\boldsymbol{\pi}}$ be another incentive-compatible escalators-and-stairs allocation with the same associated partition and that separates and pools in the same intervals as $\boldsymbol{\pi}$. If $\boldsymbol{\pi}$ separates in at least one subset Γ_i, then $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$. Otherwise, if $\boldsymbol{\pi}(\underline{\gamma}) = \hat{\boldsymbol{\pi}}(\underline{\gamma})$, then $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$.*

Proof. Suppose that there exists i such that for all $\gamma \in \Gamma_i$, $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma)$. Then $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma)$ for all $\gamma \in \Gamma'$ for any adjacent subset Γ' to Γ_i. To see this note first that if Γ' separates, then $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma) = \pi^f(\gamma)$ for all $\gamma \in \Gamma'$. If Γ' pools and Γ_i also pools, then Condition 1 part i uniquely determines the pooling action at Γ'. Finally, if Γ_i separates and Γ' pools, then Condition 1 part ii uniquely determines the pooling action at Γ'.

By induction, we can then show that if there exists i such that for all $\gamma \in \Gamma_i$, $\boldsymbol{\pi}(\gamma) = \hat{\boldsymbol{\pi}}(\gamma)$, then $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$. Now, to prove the lemma, if there is a subset where $\boldsymbol{\pi}$ separates, by construction, both allocations prescribe the same action in that subset and thus $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$. If there is no subset where $\boldsymbol{\pi}$ separates, it suffices that the prescribed actions for the lowest type are equal in both allocations. □

Before moving on to characterize sufficient conditions for an escalators-and-stairs allocation to be optimal within the set of all incentive-compatible allocations, it is helpful first to focus on optimality within the set of incentive-compatible escalators-and-stairs allocations. The above lemma tells us that it suffices to know how the allocation separates and pools in a partition (and the action assigned to the lowest type if there is no separating interval) to uniquely determine the incentive-compatible escalators-and-stairs allocation. We will use this in what follows.

4 Optimality within escalators-and-stairs

In this section, we recover the necessary conditions that must be satisfied for an escalators-and-stairs allocation to be optimal within a certain subset of all incentive-compatible escalators-and-stairs allocations.

Given an interval partition $\{\Gamma_i\}_{i=1}^n$, we define the vector $p = \{p_i\}_{i=1}^n$ to be such that $p_i = 1$ if the allocation pools in Γ_i and $p_i = 0$ otherwise. We denote as π_i the action level at which the allocation pools in Γ_i and let $\pi_1 = \boldsymbol{\pi}(\underline{\gamma})$. For a given action level π_i, we define γ_i by $\gamma_i \equiv -b'(\pi_i)$. If $\pi_i \in [\pi_f(\underline{\gamma}), \pi_f(\overline{\gamma})]$, then $\gamma_i \in \Gamma$.

Recalling that no two adjacent intervals can separate, we let

$$P_n \equiv \{p \in \{0,1\}^n | \text{ such that } p_i = p_{i+1} = 0 \text{ for no } i\}.$$

Given $\{a_i, \pi_i\}_{i=1}^n$ and a vector $p \in P_n$, Lemma 1 shows that all levels of action π_i for $i > 1$ are pinned down by incentive-compatibility constraints for a given choice of $\{a_i\}_{i=1}^n$ and π_1. Together with Condition 1, incentive-compatible escalators-and-stairs allocation are then *uniquely determined* by the following restrictions:

$$\forall i \in \{1, ..., n\} : \quad \begin{aligned} &\pi_i \in \Pi; a_i \in \Gamma; \\ &\pi_i = \pi_f(a_{i+1}), && \text{if } p_i = 1, p_{i+1} = 0; \\ &\pi_f(a_{i+1}) = \pi_{i+1}, && \text{if } p_i = 0, p_{i+1} = 1; \\ &a_{i+1}\pi_i + b(\pi_i) = a_{i+1}\pi_{i+1} + b(\pi_{i+1}), && \text{if } p_i = p_{i+1} = 1; \\ &\pi_{i+1} > \pi_i, && \text{if } p_i = p_{i+1} = 1; \\ &a_{i+1} > a_i. \end{aligned} \tag{2}$$

The constraints in the first line are just that π_i be feasible and a_i represent a valid interval bound. The second and third lines are part (ii) of Condition 1, which applies to pooling and separating regions that are adjacent. The next two lines are part (i) of Condition 1. The final line represents the restriction that the subsets of the partition be nonempty.

Given $\{a_i, \pi_i\}_{i=1}^n$ and a vector $p \in P_n$, the objective of the principal can then be written as

$$P(\{a_i, \pi_i\}_{i=1}^n | p) \equiv \sum_{i=1}^{n} \left(p_i \int_{a_i}^{a_{i+1}} w(\gamma, \pi_i) f(\gamma) d\gamma + (1 - p_i) \int_{a_i}^{a_{i+1}} w(\gamma, \pi_f(\gamma)) f(\gamma) d\gamma \right),$$

where we have re-arranged the expected payoff to the principal by first summing over the payoff in the pooling subsets, and then adding the payoff in the separating subsets.

An escalators-and-stairs allocation that is optimal within the set of all incentive-compatible allocations must also be optimal within the set of escalators-and-stairs allocations that share partitions with the same number of subsets and the same vector p. Thus, an optimal escalators-and-stairs allocation must be a solution to

$$\max_{\{a_i, \pi_i\}_{i=1}^n} P(\{a_i, \pi_i\}_{i=1}^n | p) \text{ subject to (2).} \tag{3}$$

This naturally leads to the following necessary condition:

Condition 2 (Necessary Condition). *There exists $\{\lambda_i\}_{i=1}^{n+1} \in \mathbb{R}$ such that for all $p_i = 1$ (pools):*

(i) $\int_{a_i}^{a_{i+1}} w_\pi(\gamma, \pi_i) f(\gamma) d\gamma = \lambda_{i+1}(a_{i+1} - \gamma_i) - \lambda_i(a_i - \gamma_i).$

(ii) $\lambda_{i+1} = -\left(\frac{w(a_{i+1},\pi_{i+i})-w(a_{i+1},\pi_i)}{\pi_{i+1}-\pi_i}\right) f(a_{i+1})$ *if* $p_{i+1} = 1$.

with $\lambda_1 = \lambda_{n+1} = 0$.

We can now state the following lemma.

Lemma 2. *Given* $p = \{p_i\}_{i=1}^n \in P_n$, $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ *solves Problem 3 only if* $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ *satisfies (2) and Condition 2 holds.*

Proof. In Appendix B.1. □

5 Sufficient conditions for optimality

Next, we will determine the conditions under which an allocation with interval partition $\{\Gamma_i\}_{i=1}^n$ and vector p solves the problem (1).

Similarly to Amador and Bagwell (2013), we define the constant κ as follows:

$$\kappa \equiv \inf_{(\gamma,\pi)\in\Gamma\times\Pi} \left\{ \frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)} \right\} \tag{4}$$

Denote by $\tilde{\Gamma} \equiv \{\gamma \in \Gamma | \gamma \neq \gamma_i, \forall p_i = 1\}$ as the set in which the proposed allocation does not pool at the agent's preferred action. The following function $\tilde{\Lambda} : \tilde{\Gamma} \to \mathbb{R}$ is helpful in stating the conditions.

Definition 1. *Let* $\boldsymbol{\pi}$ *be an escalators-and-stairs allocation with associated partition* $\{\Gamma_i\}_{i=1}^n$. *Let* $p = \{p_i\}_{i=1}^n \in P_n$ *be the associated pooling indicator vector and* $\{a_i, \pi_i\}_{i=1}^n$ *be the associated vector of the lower bounds of the interval and the pooling actions.*
Given $\{\lambda_i\}_{i=1}^{n+1}$, *define the function* $\tilde{\Lambda} : \tilde{\Gamma} \to \mathbb{R}$ *to be:*

$$\tilde{\Lambda}(\gamma) = \begin{cases} -w_\pi(\gamma, \pi_f(\gamma)) f(\gamma) & ; p_i = 0, \gamma \in cl(\Gamma_i) \\[2ex] \dfrac{\int_\gamma^{a_{i+1}} w_\pi(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) d\tilde{\gamma} - \lambda_{i+1}(a_{i+1} - \gamma_i)}{\gamma - \gamma_i} & ; p_i = 1, \gamma \in \Gamma_i \cap [a_i, \gamma_i). \\[2ex] \dfrac{-\int_{a_i}^\gamma w_\pi(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) d\tilde{\gamma} - \lambda_i(a_i - \gamma_i)}{\gamma - \gamma_i} & ; p_i = 1, \gamma \in \Gamma_i \cap (\gamma_i, a_{i+1}]. \end{cases} \tag{5}$$

Consider the following condition:

Condition 3 (Sufficient Condition)**.** *Given the escalators-and-stairs allocation* $\boldsymbol{\pi}$ *with interval partition* $\{\Gamma_i\}_{i=1}^{n}$ *there exists* $\{\lambda_i\}_{i=1}^{n+1}$ *with* $\lambda_1 = \lambda_{n+1} = 0$ *such that*

(c1) If $p_i = 0$*, then* $\kappa F(\gamma) + \tilde{\Lambda}(\gamma)$ *is non-decreasing* $\forall \gamma \in \Gamma_i$.

(c2) If $p_i = 1$*, then for* $\gamma \in \Gamma_i \cap [a_i, \gamma_i)$, $\kappa F(a_i) - \lambda_i \leq \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$*, with equality at* $\gamma = a_i$ *if* $a_i < \gamma_i$.

(c3) If $p_i = 1$*, then for* $\gamma \in \Gamma_i \cap (\gamma_i, a_{i+1}]$, $\kappa F(a_{i+1}) - \lambda_{i+1} \geq \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$*, with equality at* $\gamma = a_{i+1}$ *if* $\gamma_i < a_{i+1}$.

(c4) If $p_i = 1$ *and* $a_i < \gamma_i < a_{i+1}$*, then* $\kappa F(a_i) - \lambda_i \leq \kappa F(a_{i+1}) - \lambda_{i+1}$.

(c5) If $p_1 = 0$*, or* $p_1 = 1$ *and* $\gamma_1 = a_1$*, then* $w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma}) \leq 0$.

(c6) If $p_n = 0$*, or* $p_n = 1$ *and* $\gamma_n = a_{n+1}$*, then* $w_\pi(\overline{\gamma}, \pi_f(\overline{\gamma}))f(\overline{\gamma}) \geq 0$.

where $p = \{p_i\}_{i=1}^{n} \in P_n$ *is the vector associated with the pooling indicators,* $\{a_i, \pi_i\}_{i=1}^{n}$ *is the pair of vectors associated with the lower bounds of the interval and the pooling actions, and* $\tilde{\Lambda}$ *is as defined in Definition 1 given* $\{\lambda_i\}_{i=1}^{n+1}$ *.*

Condition 3 contains several parts. Part (c1) applies for intervals where the allocation separates. Parts (c2), (c3), and (c4) apply in intervals where the allocation is pooling. And parts (c5) and (c6) applies to the lowest and highest regions in the partition. With this, we can now state the main result:

Theorem 1 (Sufficiency)**.** *If an incentive compatible escalators-and-stars allocation* $\boldsymbol{\pi}$ *with associated partition* $\{\Gamma_i\}_{i=1}^{n}$ *satisfies Condition 3, then it is optimal among the class of all incentive compatible allocations.*

Proof. In Appendix B.2. □

6 Discussion of results

Theorem 1 shows that Condition 3 is sufficient for an escalators-and-stairs allocation to be optimal within the set of all incentive-compatible allocations.

6.1 Necessary conditions for optimal delegation

Given that Condition 2 is necessary for such an allocation to be optimal within a subset of incentive-compatible allocations, Condition 2 should be implied by Condition 3. We will show that this is the case, and that the values of $\{\lambda_i\}$ are the same for both. The equality condition in (c2) requires $\tilde{\Lambda}(a_i) = -\lambda_i$ for every pooling interval such that $a_i < \gamma_i$. From the expression of $\tilde{\Lambda}(\gamma)$ when $a_i < \gamma_i$, we can rewrite this equality as

$$\frac{\int_{a_i}^{a_{i+1}} w_\pi\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} - \lambda_{i+1}(a_{i+1} - \gamma_i)}{a_i - \gamma_i} = -\lambda_i \tag{6}$$

Likewise, the equality condition in (c3) implies $\tilde{\Lambda}(a_{i+1}) = -\lambda_{i+1}$ for pooling intervals such that $\gamma_i < a_{i+1}$. Using again the expression of $\tilde{\Lambda}(\gamma)$, we rewrite this equality as:

$$\frac{-\int_{a_i}^{a_{i+1}} w_\pi\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} - \lambda_i(a_i - \gamma_i)}{a_{i+1} - \gamma_i} = -\lambda_{i+1} \tag{7}$$

Isolating the integral term in (6) and (7), we have that the following must hold for every pooling interval

$$\int_{a_i}^{a_{i+1}} w_\pi\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} = \lambda_{i+1}(a_{i+1} - \gamma_i) - \lambda_i(a_i - \gamma_i) \tag{8}$$

This shows that the equality conditions of Condition 3 imply part (i) of Condition 2.

Moreover, if the optimal escalators-and-stairs allocation is such that there are two adjacent pooling intervals, then part (ii) of Condition 2 must also be implied by Condition 3.

We show in Appendix A using Conditions (c2) and (c3) that this, in turn, is satisfied provided that:

$$\frac{w_{\pi\pi}(a_{i+1}, \pi)}{b''(\pi)} = \kappa \tag{9}$$
$$\forall\, \pi \in [\pi_i, \pi_{i+1}] \text{ and } \Gamma_i, \Gamma_{i+1} \text{ s.t. } p_i = p_{i+1} = 1.$$

Note that (9) is immediately satisfied for the family of preferences with the property that $\frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)}$ is constant for all γ and π. That is, (9) is satisfied for the family of preferences such that the principal's payoff takes the form

$$w(\gamma, \pi) = A[b(\pi) + B(\gamma) + C(\gamma)\pi] \tag{10}$$

for some constant $A > 0$ and functions B and C. Following Kolotilin and Zapechelnyuk (2024), we refer to this family as the *linear family*.[4]

6.2 The conditions for interval delegation in AB

Amador and Bagwell (2013) obtained sufficient conditions for an interval allocation to be optimal. We can represent an interval allocation as an escalators-and-stairs allocation with an interval partition of at most three regions, where one region separates, the others pool, and the two regions that pool cannot be adjacent. The separating region corresponds to the interval of actions that are offered to the agent, while the pooling regions capture that actions outside the interval are not allowed.

Let us focus on the case where the interval allocation is generated by just an upper bound on the action. In this case, the partition contains just two intervals Γ_1 and Γ_2 where $p_1 = 0$ and $p_2 = 1$. In this case, $a_1 = \underline{\gamma}$, $a_3 = \overline{\gamma}$; and we let $\gamma_H = a_2$ (for comparison with Amador and Bagwell, 2013). Incentive compatibility implies $\pi_2 = \pi_f(\gamma_H)$, and it follows that $\gamma_2 = \gamma_H$. The value of $\tilde{\Lambda}$ is then:

$$\tilde{\Lambda}(\gamma) = \begin{cases} -w_\pi(\gamma, \pi_f(\gamma))f(\gamma) & ; \gamma \in [\underline{\gamma}, \gamma_H] \\ \dfrac{-\int_{\gamma_H}^{\gamma} w_\pi\left(\tilde{\gamma}, \pi_f(\gamma_H)\right) f(\tilde{\gamma}) d\tilde{\gamma}}{\gamma - \gamma_H} & ; \gamma \in (\gamma_H, \overline{\gamma}]. \end{cases} \tag{11}$$

We can now apply Condition 3 to this case. Part (c1) requires that

$$\kappa F(\gamma) - w_\pi(\gamma, \pi_f(\gamma))f(\gamma) \text{ be non-decreasing in } \gamma \in [\underline{\gamma}, \gamma_H) \tag{12}$$

Part (c5) requires that

$$w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma}) \leq 0$$

Part (c3) requires an equality condition: $\kappa - \lambda_4 = \kappa + \tilde{\Lambda}(\overline{\gamma})$. Using that $\lambda_4 = \lambda_1 = 0$, we obtain that

$$-\int_{\gamma_H}^{\overline{\gamma}} w_\pi\left(\tilde{\gamma}, \pi_f(\gamma_H)\right) f(\tilde{\gamma}) d\tilde{\gamma} = 0$$

[4] This family of preferences is identified by Amador and Bagwell (2013) and plays a special role in their analysis when they generate conditions that are also necessary (and not just sufficient) for interval delegation to be optimal. The preferences in Amador, Werning, and Angeletos (2006) fall into this class. This preference family is the one for which Kolotilin and Zapechelnyuk (2024) obtains sufficient and necessary results mapping delegation problems into persuasion problems. In their paper using majorization tools in economic applications, Kleiner, Moldovanu, and Strack (2021) also argue that their approach can be extended to the delegation problem with these same preferences (see footnote 40). Kleiner (2022) also restricts attention to this preference family (with $B(\gamma) = 0$) to characterize optimal delegation in multidimensional settings.

Using this to rewrite (c3), we get

$$\kappa(1 - F(\gamma)) \geq \tilde{\Lambda}(\gamma)$$

for $\gamma \in (\gamma_H, \overline{\gamma}]$. This becomes

$$\kappa(1 - F(\gamma)) \geq \frac{\int_\gamma^{\overline{\gamma}} w_\pi \left(\tilde{\gamma}, \pi_f(\gamma_H)\right) f(\tilde{\gamma}) d\tilde{\gamma}}{\gamma - \gamma_H} \text{ for } \gamma \in (\gamma_H, \overline{\gamma}].$$

We can summarize the equality condition with this last inequality as:

$$\kappa(\gamma - \gamma_H)(1 - F(\gamma)) \geq \int_\gamma^{\overline{\gamma}} w_\pi \left(\tilde{\gamma}, \pi_f(\gamma_H)\right) f(\tilde{\gamma}) d\tilde{\gamma} \text{ for } \gamma \in [\gamma_H, \overline{\gamma}], \tag{13}$$

with equality at $\gamma = \gamma_H$. The rest of the parts in Condition 3 do not apply.

Conditions (12) and (13) correspond to the conditions obtained in Amador and Bagwell (2013) for the optimality of a cap allocation. A similar approach can be used to show the equivalence of the conditions in Amador and Bagwell (2013) with Condition 3 for the other two interval allocation cases (i.e., the allocation that corresponds to just a lower bound on the action and the one that corresponds to both a cap and a floor). This shows that Theorem 1 contains the result of Amador and Bagwell (2013) as a special case.

6.3 The conditions for extreme delegation in MS

Melumad and Shibano (1991) solve for the optimal allocations in the case where the shocks are uniformly distributed, the preferences are linear-quadratic, and the preferred actions are linear functions of the shock. In our notation, this corresponds to $F(\gamma) = \gamma \in [0, 1]$, $b(\pi) = -\frac{\pi^2}{2}$ and $w(\gamma, \pi) = b(\pi) + (a\gamma + k)\pi$, for some $a, k \in \mathbb{R}$. This implies $\kappa = 1$, $\pi_f(\gamma) = \gamma$, and $\gamma_i = -b'(\pi_i) = \pi_i$ for $p_i = 1$.

When $a > 2$ and $k \in (1 - a, 0)$, the optimal allocation does not correspond to an interval allocation. Instead, it corresponds to a one-step allocation. In particular, the corresponding action levels, π_1 and π_2, lie outside the agent's preferred action range, that is, $\pi_1 < 0$ and $\pi_2 > 1$. We refer to this case as an *extreme one-step allocation*. We can describe an extreme one-step allocation as an escalator-and-stairs allocation with exactly two regions, $\{\Gamma_1, \Gamma_2\}$, where $p_1 = p_2 = 1$, $\pi_1 < 0$ and $\pi_2 > 1$. Incentive compatibility requires $a_2 = (\pi_1 + \pi_2)/2$.

For comparison with Melumad and Shibano (1991), let $a_1 = 0$, $a_3 = 1$, and $a_2 = \tau \in (0, 1)$.

Given an extreme one-step allocation, we can write $\tilde{\Lambda}$ in the following way:

$$\tilde{\Lambda}(\gamma) = \begin{cases} \dfrac{\gamma(2\pi_1 - 2k - a\gamma)}{2(\gamma - \pi_1)} & ; \gamma \in [0, \tau). \\ \dfrac{(1-\gamma)(a + 2k + a\gamma - 2\tau - \pi_2)}{2(\gamma - \pi_2)} & ; \gamma \in [\tau, 1]. \end{cases} \tag{14}$$

where we have already substituted for $\gamma_1 = \pi_1 < 0$ and $\gamma_2 = \pi_2 > 1$. We can now apply the relevant parts of Condition 3, (c2) and (c3). The equality condition at τ in (c2) and (c3) imply that:

$$\frac{\tau(2\pi_1 - 2k - a\tau)}{2(\tau - \pi_1)} = \frac{(1-\tau)(a + 2k + (a-2)\tau - \pi_2)}{2(\tau - \pi_2)} \tag{15}$$

Plugging the incentive-compatibility condition $\pi_2 = 2\tau - \pi_1$ and solving for π_1, we get:

$$\pi_1 = (a-2)\tau^2 + 2(1+k)\tau - k - \frac{a}{2} \tag{16}$$

Next, conditions (c2) and (c3) require that the left and right derivatives of $H(\gamma) \equiv \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$ be non-negative at $\gamma = \tau$. Thus, we have that the following must hold:

$$H'_-(\tau)(\tau - \pi_1) = -k - (a-1)\tau - \frac{\tau(2\pi_1 - 2k - a\tau)}{2(\tau - \pi_1)} \geq 0$$
$$H'_+(\tau)(\tau - \pi_2) = -k - (a-1)\tau - \frac{(1-\tau)(a + 2k + (a-2)\tau - \pi_2)}{2(\tau - \pi_2)} \leq 0$$

It follows immediately from (15) that we can reduce the above inequalities to the following equality condition:

$$k + (a-1)\tau + \frac{\tau(2\pi_1 - 2k - a\tau)}{2(\tau - \pi_1)} = 0$$

Substituting (16) into the equality above, it follows that τ must solve the following polynomial equation:

$$2(a-2)^2\tau^3 + 3(a-2)(1+2k)\tau^2 + \left(4k(k+2) - 2a(k-1) - a^2\right)\tau - k(a+2k) + 2 = 0 \tag{17}$$

Conditions (16) and (17) correspond to the necessary and sufficient conditions obtained in Melumad and Shibano (1991) for the optimality of an extreme one-step allocation.

Finally, if $a > 2$ and $k \in (1-a, 0)$, then (16) and (17) immediately imply that $\kappa F(\gamma) + \tilde{\Lambda}(\gamma)$ is continuous at $\gamma = \tau$ and nondecreasing in γ. This implies that any extreme one-step allocation that satisfies (16) and (17) also satisfies Condition 3 for $\lambda_2 = \tau(2\pi_1 - 2k - a\tau)/(2(\tau - \pi_1)) = \tilde{\Lambda}(\tau)$.

Therefore, we obtain Melumad and Shibano (1991)'s characterization of optimal extreme one-step delegation as a special case of Theorem 1.

6.4 A hybrid example

In this section, we show how our theorem can be used to prove the optimality of a hybrid allocation that contains jumps as well as regions of flexibility. For this case, we maintain the distribution from the previous example, that is, $F(\gamma) = \gamma \in [0, 1]$. We also maintain the quadratic preferences for the agent, and let $b(\pi) = -\pi^2/2$. However, we generalize the planner's preferences to be $w(\gamma, \pi) = b(\pi) + h(\gamma)\pi$. In this case, $h(\gamma)$ represents the planner's preferred action for a given γ. This specification lies within the family of cases considered in Alonso and Matouschek (2008).[5]

For this example, we specialize the planner's preferred choice to the following function (symmetric around $1/2$):

$$h(\gamma) = \begin{cases} 0 & \text{for } 0 \leq \gamma < \frac{1}{2} - \epsilon, \\ \left(\gamma - \frac{1}{2} + \epsilon\right) \frac{1}{2\epsilon} & \text{for } \gamma \in \left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right), \\ 1 & \text{for } \frac{1}{2} + \epsilon \leq \gamma \leq 1, \end{cases}$$

for some value of $\epsilon \in (0, 1/4)$.

We will look for conditions where the solution to the optimal delegation problem takes the form of an exclusion region in the center of the type distribution. Anticipating symmetry, the proposed allocation takes the form:

$$\hat{\pi}(\gamma|a_2) = \begin{cases} \gamma & \text{for } 0 \leq \gamma < a_2 \\ a_2 & \text{for } a_2 \leq \gamma < 1/2 \\ 1 - a_2 & \text{for } 1/2 \leq \gamma < 1 - a_2 \\ \gamma & \text{for } 1 - a_2 \leq \gamma < 1 \end{cases} \tag{18}$$

for some value of $a_2 \in (0, 1/2 - \epsilon)$. Note that the above allocation is incentive compatible, as it is symmetric around the agent's preferred action at the jump at $1/2$. We can plug this allocation into the planner's objective, and optimize for a_2. Solving the resulting first-order condition, we obtain that:

$$a_2^\star = \frac{1}{4}\left(1 - \sqrt{1 - 4\epsilon}\right)$$

[5] Alonso and Matouschek (2008) in Result 6 Section 7 (and their Figure 8) present a very similar example to the one in this section, but where the principal's preferences are linearly increasing over the full support (rather than constant at the extremes as in ours). We choose a different example because it highlights how our methods can be used in non-linear settings. In addition, as we discuss at the end of this subsection, our example can be extended to one where the optimal allocation features a countable number of jumps.

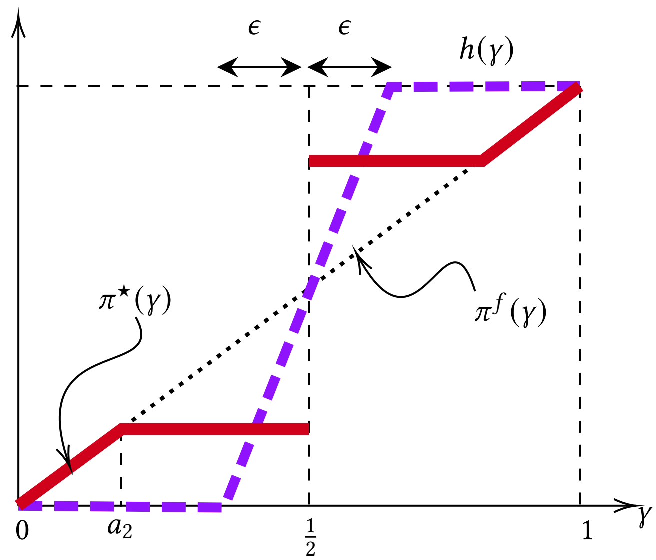

Figure 2: The (symmetric) hybrid example.

which is indeed in $(0, 1/2 - \epsilon)$.

The proposed allocation is $\pi^\star(\gamma) = \hat{\pi}(\gamma|a_2^\star)$ for all $\gamma \in \Gamma$. Its associated interval partition is composed of four intervals with boundaries given by $a_1 = 0, a_2 = \frac{1}{4}(1 - \sqrt{1 - 4\epsilon}), a_3 = 1/2, a_4 = 1 - a_2, a_5 = 1$; with $p_1 = 0, p_2 = 1, p_3 = 1, p_4 = 0$. In addition, $\pi_2 = a_2$ and $\pi_3 = a_4$, while $\gamma_2 = a_2$ and $\gamma_3 = a_4$. Figure 2 displays the proposed allocation, together with the agent's and the principal's preferred choices.

We can now use Definition 1 to construct $\tilde{\Lambda}$. For $\gamma \in [0, a_2]$, $\tilde{\Lambda}(\gamma) = -w_\pi(\gamma, \pi_f(\gamma))f(\gamma) = \gamma$. For $\gamma \in [a_4, 1]$ we have that $\tilde{\Lambda}(\gamma) = -w_\pi(\gamma, \pi_f(\gamma))f(\gamma) = -1 + \gamma$. For $\gamma \in (a_2, a_3]$, given that $\gamma_2 = a_2$, we have that

$$\tilde{\Lambda}(\gamma) = \frac{-\int_{a_2}^{\gamma} w_\pi(\tilde{\gamma}, \pi_2)f(\tilde{\gamma}) - \lambda_2(a_2 - \gamma_2)}{\gamma - a_2}$$

This becomes:

$$\tilde{\Lambda}(\gamma) = \begin{cases} \dfrac{1}{4}(1 - \sqrt{1 - 4\epsilon}) & \text{for } a_2 \leq \gamma < 1/2 - \epsilon \\ \dfrac{(1 - 2\gamma)(1 - 2\gamma - 2\epsilon(1 + \sqrt{1 - 4\epsilon}))}{4(1 - 4\gamma - \sqrt{1 - 4\epsilon})} & \text{for } 1/2 - \epsilon \leq \gamma < 1/2 \end{cases}$$

Finally, for $\gamma \in [a_3, a_4)$, given that $\gamma_3 = a_4$, we have that

$$\tilde{\Lambda}(\gamma) = \frac{\int_{\gamma}^{a_4} w_\pi(\tilde{\gamma}, \pi_3)f(\tilde{\gamma}) - \lambda_4(a_4 - \gamma_3)}{\gamma - a_3}$$

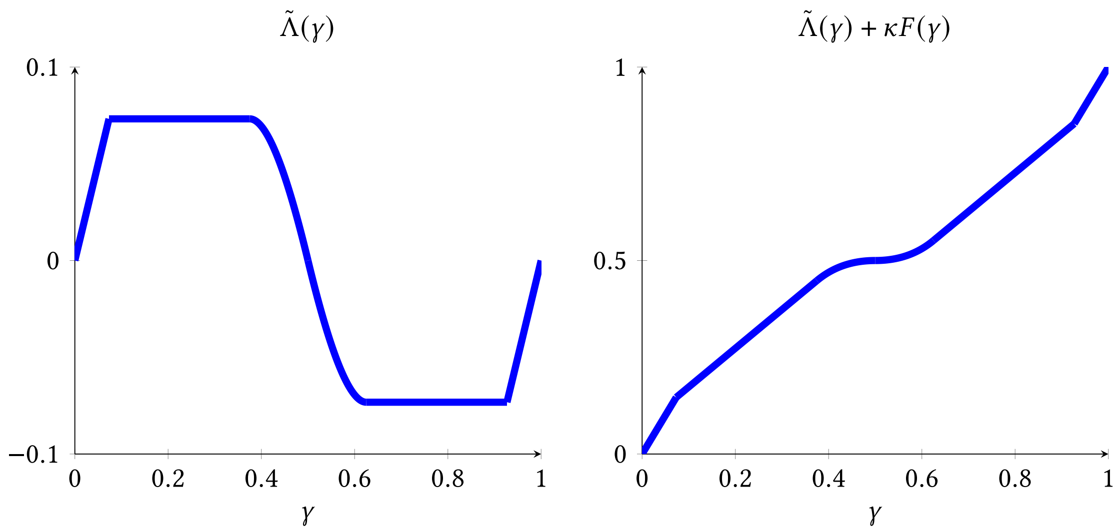

Figure 3: The function $\tilde{\Lambda}(\gamma)$ for the hybrid example.
Note: The figure is plotted assuming $\epsilon = 1/8$.

which delivers

$$\tilde{\Lambda}(\gamma) = \begin{cases} \dfrac{-(1-2\gamma)(1-2\gamma+2\epsilon(1+\sqrt{1-4\epsilon}))}{4(3-4\gamma+\sqrt{1-4\epsilon})} & \text{for } 1/2 \leq \gamma < 1/2+\epsilon \\ \dfrac{1}{4}(\sqrt{1-4\epsilon}-1) & \text{for } 1/2+\epsilon \leq \gamma < a_4, \end{cases}$$

and this completes the characterization of $\tilde{\Lambda}$. Figure 3 displays the shape of the multiplier for a given value of ϵ ($\epsilon = 1/8$).

We are now ready to check the Condition 3. First, note that $\kappa = 1$. Part (c1) is satisfied as $\kappa F(\gamma) + \tilde{\Lambda}(\gamma)$ is nondecreasing in $[0, a_2)$ and $[a_4, 1]$. For part (c2), the interval that applies is $[a_3, a_4)$ with $\gamma_3 = a_4$. The equality condition requires that

$$a_3 - \lambda_3 = a_3 + \tilde{\Lambda}(a_3) \Rightarrow \lambda_3 = 0$$

One can also show that $\tilde{\Lambda}(\gamma) + \kappa F(\gamma)$ is non-decreasing for $\gamma \in [a_3, a_4)$.[6] Part (c3) is similar but applies to $\gamma \in (a_2, a3]$ with $\gamma_2 = a_2$. The equality condition is satisfied with $\lambda_3 = 0$, and a similar argument shows that $\tilde{\Lambda}$ is nondecreasing in this region. Condition (c4) does not apply. The final conditions (c5) and (c6) hold as $w_\pi(\gamma, \pi_f(\gamma)) = 0$ for $\gamma \in \{0, 1\}$. Given this, we can apply Theorem

[6] For $\gamma \geq 1/2+\epsilon$, $\Lambda(\gamma)$ is constant, so the result in that range is immediate. For $\gamma \in [1/2, 1/2+\epsilon)$, we have that

$$\frac{d(\Lambda(\gamma)+\gamma)}{d\gamma} = (2\gamma-1)\frac{2-2\gamma+\sqrt{1-4\epsilon}}{(3-4\gamma+\sqrt{1-4\epsilon})^2}\frac{1-4\epsilon}{\epsilon} \geq 0$$

for $\gamma \in [1/2, 1]$. Finally, $\tilde{\Lambda}(\gamma)$ being continuous in $[a_3, a_4)$ guarantees that $\tilde{\Lambda}(\gamma)+\gamma$ is not decreasing.

1 to show that the proposed allocation solves the delegation problem.

An interesting feature of this case is that it can easily be extended to study the optimality of an allocation with multiple jumps. For example, suppose that the support is now $[0,2]$. Suppose that the agent's preferences remain linear in γ. The preferred choice of the principal is now given by $h_e(\gamma)$ where $h_e(\gamma) = h(\gamma)$ for $\gamma \in [0,1]$ and $h_e(\gamma) = h(\gamma-1)+1$ for $\gamma \in [1,2]$. Basically, we have "expanded" the principal preferences by repeating the same pattern of $[0,1]$ over $[1,2]$ but shifted up so that the principal's preferred choice remains continuous. Consider then the similarly "shifted and expanded" proposed allocation $\pi_e^\star(\gamma)$ where $\pi_e^\star(\gamma) = \pi^\star(\gamma)$ for $\gamma \in [0,1]$ and $\pi_e^\star(\gamma) = \pi^\star(\gamma-1)+1$ for $\gamma \in [1,2]$. This proposed allocation features two jumps (one at $1/2$ as before and another at $3/2$). We can replicate the steps above by constructing $\tilde{\Lambda}$ for each subinterval $[0,1]$ and $[1,2]$ and using Theorem 3 to prove the optimality of the proposed allocation. In this fashion, we can continue to expand the preferences and the preferred allocation, and thus incorporate arbitrarily many jumps into the analysis.

6.5 The hybrid example outside the linear family

We now consider an extension of the hybrid example, where the principal's payoff is given by $w(\pi,\gamma) = A(\gamma)(b(\pi) + h(\gamma)\pi)$, where b and h are as defined in Section 6.4. This example falls outside the linear family in (10) by allowing the relative risk aversion between the principal and the agent to depend on the state. In particular, the relative risk aversion is given by:

$$A(\gamma) = \frac{w_{\pi\pi}(\pi,\gamma)}{b_{\pi\pi}(\pi,\gamma)}$$

Note that if $A(\gamma)$ is constant and equal to 1, we recover the example from the previous section.

The goal of this extension is to demonstrate that our conditions readily extend beyond the family in (10). For this example, we consider the following expression for $A(\gamma)$. Let

$$\alpha \in \left(0, \frac{1}{4}\left(1 - \sqrt{1-4\epsilon}\right)\right],$$

for $\epsilon \in (0, 1/4)$. Then, $A(\gamma)$ is defined as:

$$A(\gamma) = \begin{cases} 2 - \frac{\gamma}{\alpha}, & \text{for } \gamma \in [0,\alpha] \\ 1, & \text{for } \gamma \in [\alpha, 1-\alpha] \\ 2 - \frac{1-\gamma}{\alpha}, & \text{for } \gamma \in [1-\alpha, 1] \end{cases}$$

Figure 4 displays the relative risk aversion between the principal and the agent. We propose

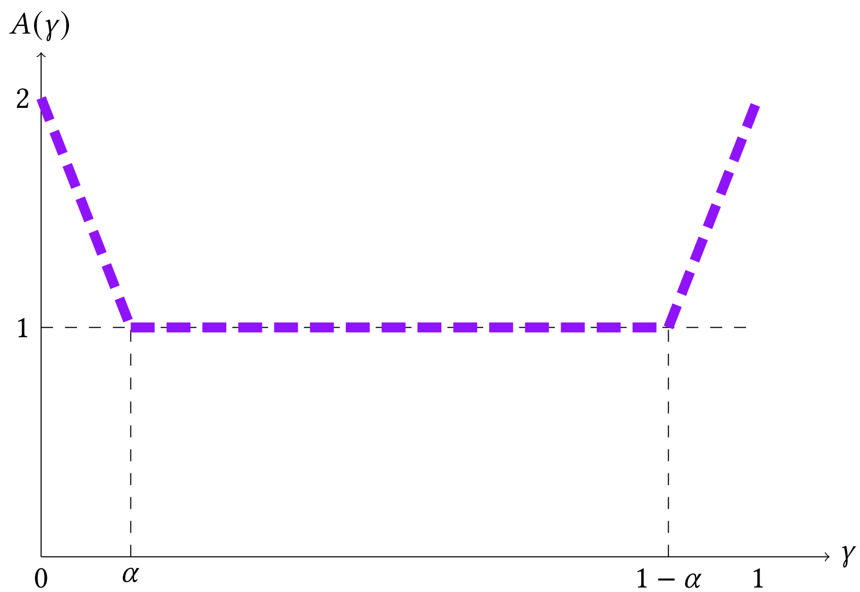

Figure 4: Example beyond the family in (10).

the same hybrid allocation as before, which takes the form of equation (18) with $a_2^* = \frac{1}{4}\big(1 - \sqrt{1-4\epsilon}\big)$.

We are now ready to check Condition 3. First, note that $\kappa = min_{\gamma\in[0,1]}A(\gamma) = 1$. Moreover, $A(\gamma) = 1$ for $\gamma \in (a_2^*, 1-a_2^*)$. That is, both κ and the relative risk aversion $A(\gamma)$ in the pooling intervals are the same as before. Therefore, it is immediate that conditions (c2) and (c3) still hold for $\lambda_3 = 0$.

It remains to check conditions (c1), (c5) and (c6), which apply to the separating interval. We can write the expressions for $\tilde{\Lambda}(\gamma) = -w_\pi(\gamma, \pi_f(\gamma))f(\gamma)$ as follows:

$$\begin{cases} \gamma\left(2-\frac{\gamma}{\alpha}\right) & \text{for } \gamma \le \alpha \\ \gamma & \text{for } \alpha < \gamma \le a_2^* \\ (\gamma-1) & \text{for } 1-a_2^* \le \gamma < 1-\alpha \\ (\gamma-1)\left(2-\frac{1-\gamma}{\alpha}\right) & \text{for } 1-\alpha \le \gamma \end{cases}$$

Note that $-w_\pi(\gamma, \pi_f(\gamma))f(\gamma) = 0$ for $\gamma \in \{0, 1\}$ and, thus, (c5) and (c6) hold. Moreover, it is straightforward to check that $\tilde{\Lambda}(\gamma) + \kappa F(\gamma)$ is increasing in γ for $\gamma < a_2^*$ and $\gamma > 1 - a_2^*$. Therefore, our proposed allocation is optimal.

This example illustrates the more general principle that our framework is robust to certain variations in the principal's risk preferences. Specifically, increasing the principal's risk aversion in separating regions of an optimal allocation does not alter the optimality of that allocation, as long as the monotonicity condition in (c1) is maintained. This robustness highlights the flexibility and applicability of Condition 3, extending its relevance beyond the family of preferences described by (10).

7 Extension to Money Burning

In this section, we extend our sufficient condition to the case where money burning is feasible, allowing us to analyze the optimality of escalators-and-stairs allocations under these conditions. We show that the sufficient conditions derived in the previous sections can be adjusted to check for the optimality of escalators-and-stairs allocations that burn no money.

First, let $\boldsymbol{t}(\gamma)$ denote the money that is burned in state γ, which generates equal losses for the principal and agent. The principal's problem is to choose an allocation $(\boldsymbol{\pi}, \boldsymbol{t})$, which is a pair of functions that determine the action and the amount of money burned given the private information of the agent. The objective is to choose $(\boldsymbol{\pi}, \boldsymbol{t})$ to maximize the principal's welfare function subject to the incentive constraints of the agent. The delegation problem with money burning can be expressed as follows:

$$
\begin{aligned}
&\max_{\{\boldsymbol{\pi},\boldsymbol{t}:\Gamma\to\Pi\}} \int_{\Gamma} (w(\gamma, \boldsymbol{\pi}(\gamma)) - \boldsymbol{t}(\gamma)) f(\gamma) d\gamma \quad \text{subject to:} \\
&\qquad \gamma \in \arg\max_{\tilde{\gamma}\in\Gamma} \{\gamma \boldsymbol{\pi}(\tilde{\gamma}) + b(\boldsymbol{\pi}(\tilde{\gamma})) - \boldsymbol{t}(\tilde{\gamma})\} \\
&\qquad \boldsymbol{t}(\gamma) \geq 0, \forall \gamma \in \Gamma
\end{aligned}
\tag{19}
$$

Note that Problem 1 is equivalent to Problem 19 with the added constraint that $\boldsymbol{t}(\gamma) = 0$ for all γ. In particular, any incentive compatible $\boldsymbol{\pi}$ in the original problem is also incentive compatible in the problem with money burning if it does not burn money in all states. We refer to an incentive compatible escalator-and-stairs allocation $(\boldsymbol{\pi}, \boldsymbol{t})$ as any allocation such that $\boldsymbol{\pi}$ is within the escalator-and-stairs class and satisfies Condition 1, and $\boldsymbol{t}(\gamma) = 0$ for all γ.

Following Amador and Bagwell (2013), we redefine κ in the following way:

$$
\kappa = \min\left\{ \inf_{(\gamma,\pi)\in\Gamma\times\Pi} \frac{w_{\pi\pi}(\gamma, \pi)}{b''(\pi)}, 1 \right\}. \tag{20}
$$

With this, we can extend our sufficient condition to the delegation problem with money burning.

Theorem 2 (Sufficiency with Money Burning)**.** *If an incentive-compatible escalators-and-stairs allocation $\boldsymbol{\pi}$ with associated partition $\{\Gamma_i\}_{i=1}^n$ satisfies Condition 3 with κ given by (20) , then it is optimal among the class of all incentive-compatible allocations.*

Sketch of a proof. Proof from Amador and Bagwell (2013).

Step 1: Define the Lagrangian.

We use the integral form of the IC constraint, solve for the money-burning term $\boldsymbol{t}(\underline{\gamma})$ and substitute it into the objective function. Setting up the Lagrangian and integrating by parts yields:

$$\mathcal{L}(\boldsymbol{\pi}, \boldsymbol{t}(\underline{\gamma}) \mid \hat{\Lambda}) \equiv \int_{\gamma\in\Gamma} \Big(v(\gamma, \boldsymbol{\pi}(\gamma)) f(\gamma) + (\hat{\Lambda}(\gamma) - F(\gamma))\boldsymbol{\pi}(\gamma) \Big)\, d\gamma$$

$$+ \int_{\gamma\in\Gamma} (\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))\, d\hat{\Lambda}(\gamma) + \hat{\Lambda}(\underline{\gamma})\underline{U}.$$

where $v(\gamma, \boldsymbol{\pi}(\gamma)) = w(\gamma, \boldsymbol{\pi}(\gamma)) - \gamma\boldsymbol{\pi}(\gamma) - b(\boldsymbol{\pi}(\gamma))$; $\underline{U} = \underline{\gamma}\boldsymbol{\pi}(\underline{\gamma}) - b(\boldsymbol{\pi}(\underline{\gamma})) - \boldsymbol{t}(\underline{\gamma})$ and $\hat{\Lambda}(\gamma)$ is the (cumulative) Lagrange multiplier associated with the $\boldsymbol{t}(\gamma) \geq 0$ constraint. It is required that $\hat{\Lambda}$ is non-decreasing in γ.

Step 2: Monotonicity of the Lagrangian.

We propose the multiplier $\hat{\Lambda}(\gamma) = \Lambda(\gamma) + \kappa F(\gamma) + (1-\kappa)F(\gamma)$, where $\Lambda(\gamma)$ is the proposed multiplier for the case without money burning (Theorem 1). Condition 3 ensures that $\Lambda(\gamma) + \kappa F(\gamma)$ is non-decreasing in γ, while $\kappa < 1$, implies that $(1-\kappa)F(\gamma)$ is also non-decreasing.

Step 3: Concavity of the Lagrangian.

Rewrite the Lagrangian as

$$\mathcal{L}(\boldsymbol{\pi}, \boldsymbol{t}(\underline{\gamma}) \mid \Lambda) = \int_{\Gamma} [v(\gamma, \boldsymbol{\pi}(\gamma)) - (\kappa - 1)(\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))] f(\gamma) + (\hat{\Lambda}(\gamma) - F(\gamma))\boldsymbol{\pi}(\gamma) d\gamma$$
$$+ \int_{\Gamma} (\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma))) d((\kappa - 1)F(\gamma) + \hat{\Lambda}(\gamma))$$

where we use the fact that $\hat{\Lambda}(\underline{\gamma}) = 0$, which follows from $\Lambda(\underline{\gamma}) = 0$. The term in the fist integral is concave given the definition of κ. For the second term, note that $(\kappa - 1)F(\gamma) + \hat{\Lambda}(\gamma)$ is non-decreasing given Condition 3 which is needed in the second integral to guarantee that the concavity of b is not reversed.

Step 4: Maximize the Lagrangian.

That the Lagrangian is maximized at the proposed allocation is similar to the argument used in Theorem 1. To see this, first note that $\boldsymbol{t}(\underline{\gamma})$ does not appear in the Lagrangian, given the proposed Lagrange multiplier. Using the fact that $\Lambda(\gamma) = \hat{\Lambda}(\gamma) - F(\gamma)$, we can rewrite the Lagrangian in a equivalent way to the Lagrangian in the proof of the no money-burning case with κ given by equation (20). □

Both Kolotilin and Zapechelnyuk (2024) and Kleiner, Moldovanu, and Strack (2021) study the case of stochastic mechanisms, which is equivalent to allowing for money-burning when restricting attention to the family of preferences in (10) when $A = 1$. The hybrid example in Section 6.5 shows that our results are more general, as they include situations where money burning is strictly less costly to the planner than introducing randomness in the allocation (for the example in Section 6.5, this occurs in the regions where $A(\gamma) > 1$).

8 Conclusion

We have provided a sufficient theorem for solving the delegation problem. Its conditions map to known results for interval allocations, while crucially extending them to the cases of discontinuous allocations. We have also provided examples that illustrate the versatility of our approach, both supporting existing understanding and shedding light into new optimal scenarios.

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A Necessary conditions for Optimal Delegation

Here we show that if the optimal escalators-and-stairs allocation is such that there are two adjacent pooling intervals, then Condition 3 implies part (ii) of Condition 2 if and only if:

$$\frac{w_{\pi\pi}(a_{i+1},\pi)}{b''(\pi)} = \kappa$$
$$\forall\, \pi \in [\pi_i, \pi_{i+1}] \text{ and } \Gamma_i, \Gamma_{i+1} \text{ s.t. } p_i = p_{i+1} = 1.$$

Let Γ_i, Γ_{i+1} be such that $p_i = p_{i+1} = 1$. Note that this implies $\gamma_i < a_{i+1} < \gamma_{i+1}$. The equality conditions in (c3) for Γ_i and (c2) for Γ_{i+1} imply that $\tilde{\Lambda}(\gamma)$ is continuous at $\gamma = a_{i+1}$ and $\tilde{\Lambda}(a_{i+1}) = -\lambda_{i+1}$. Next, consider the function $H(\gamma) \equiv \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$. Conditions (c2) and (c3) require that the left and right derivative of $H(\gamma) \equiv \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$ are non-negative at $\gamma = a_{i+1}$. Differentiating $H(\gamma)$ from the left gives us:

$$H'_{-}(a_{i+1}) = \kappa f(a_{i+1}) - \frac{w_\pi(a_{i+1},\pi_i) f(a_{i+1})}{(a_{i+1}-\gamma_i)} + \frac{\tilde{\Lambda}(a_{i+1})}{(a_{i+1}-\gamma_i)} \tag{21}$$

Multiplying both sides by $(a_{i+1} - \gamma_i) \geq 0$ we get the following condition:

$$H'_{-}(a_{i+1})(a_{i+1}-\gamma_i) = (a_{i+1}-\gamma_i)\kappa f(a_{i+1}) - w_\pi(a_{i+1},\pi_i) f(a_{i+1}) - \lambda_{i+1} \geq 0 \tag{22}$$

An analogous calculation gives us the following condition for the right derivative of $H(\gamma)$:

$$H'_{+}(a_{i+1})(a_{i+1}-\gamma_{i+1}) = (a_{i+1}-\gamma_{i+1})\kappa f(a_{i+1}) - w_\pi(a_{i+1},\pi_{i+1}) f(a_{i+1}) - \lambda_{i+1} \leq 0 \tag{23}$$

Rearranging terms, we can write (22) and (23) as:

$$\lambda_{i+1} \leq -[w_\pi(a_{i+1},\pi_i) - (a_{i+1}-\gamma_i)\kappa] f(a_{i+1}) \tag{24}$$
$$\lambda_{i+1} \geq -[w_\pi(a_{i+1},\pi_{i+1}) - (a_{i+1}-\gamma_{i+1})\kappa] f(a_{i+1}) \tag{25}$$

Next, consider the function $m(\pi) = -[w_\pi(a_{i+1},\pi) - (a_{i+1} + b'(\pi))\kappa]$. Differentiating $m(.)$, we can show that $m(.)$ is non-decreasing:

$$m'(\pi) = -b''(\pi)\left[\frac{w_{\pi\pi}(a_{i+1},\pi)}{b''(\pi)} - \kappa\right] \geq 0 \tag{26}$$

where the inequality follows from $b''(\pi) < 0$ and $\kappa \leq \frac{w_{\pi\pi}(\gamma,\pi)}{b''(\pi)}$ for all γ and π. Therefore, $m(\pi_i) \leq m(\pi_{i+1})$. Assume that $m(\pi_i) < m(\pi_{i+1})$. Recall that $b'(\pi_i) = -\gamma_i$ and $b'(\pi_{i+1}) = -\gamma_{i+1}$. Therefore, we have $-[w_\pi(a_{i+1}, \pi_i) - (a_{i+1} - \gamma_i)\kappa] < -[w_\pi(a_{i+1}, \pi_{i+1}) - (a_{i+1} - \gamma_{i+1})\kappa]$ and, thus, (24) and (25) cannot be satisfied simultaneously for any $\lambda_{i+1} \in \mathbb{R}$.

Note then that there exists $\lambda_{i+1} \in \mathbb{R}$ that satisfies (24) and (25) if and only if $m(\pi_i) = m(\pi_{i+1})$. Since $m(.)$ is non-decreasing, this is equivalent to $m(\pi)$ constant for $\pi \in [\pi_i, \pi_{i+1}]$. From the expression of $m'(\pi)$, this holds if and only if $\frac{w_{\pi\pi}(a_{i+1},\pi)}{b''(\pi)} = \kappa$ for all $\pi \in [\pi_i, \pi_{i+1}]$.

Therefore, assuming that $m(\pi)$ constant for $\pi \in [\pi_i, \pi_{i+1}]$, Condition 3 implies that $\lambda_{i+1} = m(\pi)f(a_{i+1})$ for every $\pi \in [\pi_i, \pi_{i+1}]$. Integrating both sides yields:

$$\lambda_{i+1}(\pi_{i+1} - \pi_i) = \left(\int_{\pi_i}^{\pi_{i+1}} m(\pi)\right) f(a_{i+1}) \tag{27}$$

To show that the above equality is equivalent to part (ii) of Condition 2, suffices to show that $\int_{\pi_i}^{\pi_{i+1}} m(\pi) = w(a_{i+1}, \pi_i) - w(a_{i+1}, \pi_{i+1})$.

Integrating $m(.)$ over the interval $[\pi_i, \pi_{i+1}]$ yields:

$$\int_{\pi_i}^{\pi_{i+1}} m(\pi) = w(a_{i+1}, \pi_i) - w(a_{i+1}, \pi_{i+1}) - \kappa(a_{i+1}(\pi_{i+1} - \pi_i) + b(\pi_{i+1}) - b(\pi_i)) \tag{28}$$

Incentive compatibility requires that $a_{i+1}\pi_{i+1} + b(\pi_{i+1}) = a_{i+1}\pi_i + b(\pi_i)$. Therefore, the second term in the right hand side of the equation above is zero, which concludes the proof.

B Proofs

B.1 Proof of Lemma 2

Proof. Given $p = \{p_i\}_{i=1}^n \in P_n$, let $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ be a solution to Problem 1. Define ϵ to be such that

$$0 < \epsilon < \min_{i \in \{1,\ldots,n\}} \{a_{i+1}^\star - a_i^\star\}$$

Such a value ϵ exists given that $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ satisfies (2) given p. Now, consider the following version of Problem 1:

$$\max_{\{a_i,\pi_i\}_{i=1}^n} \sum_{i=1}^n \left(p_i \int_{a_i}^{a_{i+1}} w(\gamma,\pi_i)f(\gamma)d\gamma + (1-p_i)\int_{a_i}^{a_{i+1}} w(\gamma,\pi_f(\gamma))f(\gamma)d\gamma\right) \tag{29}$$

subject to:

$$\begin{aligned}
&a_{i+1}\pi_i + b(\pi_i) = a_{i+1}\pi_{i+1} + b(\pi_{i+1}), && \text{if } p_i = p_{i+1} = 1;\\
&\pi_i = \pi_f(a_{i+1}), && \text{if } p_i = 1, p_{i+1} = 0;\\
&\pi_f(a_{i+1}) = \pi_{i+1}, && \text{if } p_i = 0, p_{i+1} = 1;\\
&\pi_{i+1} \geq \pi_i, a_{i+1} \geq a_i + \epsilon, \pi_i \in \Pi, a_i \in \Gamma, \forall i \in \{1,\ldots,n\}.
\end{aligned}$$

where we have replaced the constraint $a_{i+1} > a_i$ with the stricter (but closed) constraint $a_{i+1} \geq a_i + \epsilon$. If $\{a_i^\star, \pi_i^\star\}_{i=1}^n$ solves Problem 1, then it must also solve this new Problem 29 as every allocation in the constraint set of Problem 29 is also in 1. We will use this new problem to derive necessary conditions.

Let i be such that $p_i = 1$. We consider the following cases.

First, consider the case that $p_{i+1} = 0$ for $i \leq n-1$. From the constraints, we then have that $\pi_i^\star = \pi_f(a_{i+1}^\star)$. Moreover, note that, if $i \geq 2$, then $p_{i-1} = 1$ (as $p \in P_n$). Setting $\pi_i^\star = \pi_f(a_{i+1}^\star)$ in the constraint corresponding to $p_{i-1} = p_i = 1$ yields that $a_i^\star \pi_{i-1}^\star + b(\pi_{i-1}^\star) = a_i^\star \pi_f(a_{i+1}^\star) + b(\pi_f(a_{i+1}^\star))$. Let λ_i be the Lagrange multiplier associated with this constraint (if $i = 1$, then simply set $\lambda_1 = 0$). Moreover, we plug $\pi_i^\star = \pi_f(a_{i+1}^\star)$ into the objective function. Note that $a_{i+1}^\star$ shows up in the expression for the objective function only in the following term:

$$\int_{a_i^\star}^{a_{i+1}^\star} w(\gamma,\pi_f(a_{i+1}^\star))f(\gamma)d\gamma + \int_{a_{i+1}^\star}^{a_{i+2}^\star} w(\gamma,\pi_f(\gamma))f(\gamma)d\gamma$$

The FOC with respect to a_{i+1} then yields the condition:

$$\left(\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma,\pi_f(a_{i+1}^\star))f(\gamma)d\gamma\right)\pi_f'(a_{i+1}^\star) = -\lambda_i\left[a_i^\star + b'(\pi_f(a_{i+1}^\star)\right]\pi_f'(a_{i+1}^\star)$$

Using that $\pi_f(a_{i+1}^\star) = \pi_i^\star$, $\pi_f'(a_{i+1}^\star) > 0$ and $b'(\pi_f(a_{i+1}^\star)) = -a_{i+1}^\star$, gives us:

$$\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma,\pi_i^\star)f(\gamma)d\gamma = -\lambda_i(a_i^\star - a_{i+1}^\star) \tag{30}$$

Finally, from the definition of γ_i, it follows that $\gamma_i = a_{i+1}$. We can thus write the above equality as:

$$\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma, \pi_i^\star) f(\gamma) d\gamma = \lambda_{i+1}(a_{i+1}^\star - \gamma_i) - \lambda_i(a_i^\star - \gamma_i),$$

for an arbitrary λ_{i+1}. This implies that condition (i) is necessary.

For $p_{i-1} = 0$ and $i \geq 2$ a similar argument delivers the proof for the necessity of condition (i), so we omit it.

Next, assume that $p_i = 1$ and there is no adjacent separating interval. Then it must be that $p_{i-1} = 1$ for $i \geq 2$ and $p_{i+1} = 1$ for $i \leq n-1$. From the constraints, we then have that $a_i^\star \pi_{i-1}^\star + b(\pi_{i-1}^\star) = a_i^\star \pi_i^\star + b(\pi_i^\star)$ for $i \geq 2$ and $a_{i+1}^\star \pi_i^\star + b(\pi_i^\star) = a_{i+1}^\star \pi_{i+1}^\star + b(\pi_{i+1}^\star)$ for $i \leq n-1$. Let λ_i be the Lagrange multiplier associated with the first constraint (setting $\lambda_i = 0$ if $i = 1$) and λ_{i+1} be the Lagrange multiplier associated with the second constraint (setting $\lambda_{n+1} = 0$ if $i = n$). Note that $\pi_i^\star$ shows up in both constraints, while a_{i+1}^* appers in the latter. Moreover, $\pi_i^\star$ and a_{i+1}^* also appear in the following terms in the objective function:

$$\int_{a_i^\star}^{a_{i+1}^\star} w(\gamma, \pi_i^\star) f(\gamma) d\gamma + \int_{a_{i+1}^\star}^{a_{i+2}^\star} w(\gamma, \pi_{i+1}^\star) f(\gamma) d\gamma$$

Taking the FOC with respect to $\pi_i^\star$ yields the following condition:

$$\int_{a_i^\star}^{a_{i+1}^\star} w_\pi(\gamma, \pi_i^\star) f(\gamma) d\gamma = \lambda_{i+1}(a_{i+1}^\star + b'(\pi_i^\star)) - \lambda_i(a_i^\star + b'(\pi_i^\star)) \tag{31}$$

Using that $\gamma_i = -b'(\pi_i^\star)$ implies that (i) is necessary. Taking the FOC with respect to $a_{i+i}^\star$ yields:

$$(w(a_{i+1}^*, \pi_i^\star) - w(a_{i+1}^*, \pi_{i+1}^\star)) f(a_{i+1}) = -\lambda_{i+1}(\pi_i^\star - \pi_{i+1}^\star) \tag{32}$$

This shows that (ii) is necessary, which concludes the proof. □

B.2 Proof of Theorem 1

Proof. We prove each part of this proposition separately. For all cases, let $\boldsymbol{\pi}^* : \Gamma \to \Pi$ denote the proposed interval allocation with interval partition $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ such that Condition 3 is satisfied.

By writing the incentive constraints in their usual integral form plus a monotonicity restriction, we can rewrite Problem 1 as:

$$\max_{\{\boldsymbol{\pi}:\Gamma\to\Pi\}} \int_{\Gamma} w(\gamma,\boldsymbol{\pi}(\gamma))f(\gamma)d\gamma \quad \text{subject to:} \tag{33}$$

$$\gamma\boldsymbol{\pi}(\gamma)+b(\boldsymbol{\pi}(\gamma)) = \int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})d\tilde{\gamma}+\underline{U} \quad \forall\gamma\in\Gamma \tag{34}$$

$$\boldsymbol{\pi} \text{ non-decreasing} \tag{35}$$

where $\underline{U}=\underline{\gamma}\boldsymbol{\pi}(\underline{\gamma})+b(\boldsymbol{\pi}(\underline{\gamma}))$.

Following Amador and Bagwell (2013), we first embed the monotonicity constraint (35) into the choice set of $\boldsymbol{\pi}$. We then, write constraint (34) as two inequalities:

$$\int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})d\tilde{\gamma}+\underline{U}-\gamma\boldsymbol{\pi}(\gamma)-b(\boldsymbol{\pi}(\gamma)) \leq 0 \quad \forall\gamma\in\Gamma \tag{36}$$

$$-\int_{\underline{\gamma}}^{\gamma} \boldsymbol{\pi}(\tilde{\gamma})d\tilde{\gamma}-\underline{U}+\gamma\boldsymbol{\pi}(\gamma)+b(\boldsymbol{\pi}(\gamma)) \leq 0 \quad \forall\gamma\in\Gamma \tag{37}$$

By assigning cumulative Lagrange multiplier functions Λ_1 and Λ_2 to constraints (36) and (37), respectively, we can write the Lagrangian for the problem:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\pi}\mid\Lambda_1,\Lambda_2) \equiv& \int_{\Gamma} w(\gamma,\boldsymbol{\pi}(\gamma))dF(\gamma)\\ &-\int_{\Gamma}\left(\int_{\underline{\gamma}}^{\gamma} \pi(\gamma')\,d\gamma'+\underline{U}-\gamma\pi(\gamma)-b(\boldsymbol{\pi}(\gamma))\right)d\left(\Lambda_1(\gamma)-\Lambda_2(\gamma)\right).\end{aligned}$$

The Lagrange multipliers Λ_1 and Λ_2 are restricted to be nondecreasing functions. Let $\Lambda(\gamma)\equiv\Lambda_1(\gamma)-\Lambda_2(\gamma)$. Integrating by parts the Lagrangian, we get:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\pi}\mid\Lambda) =& \int_{\Gamma}[w(\gamma,\boldsymbol{\pi}(\gamma))f(\gamma)-(\Lambda(\overline{\gamma})-\Lambda(\gamma))\boldsymbol{\pi}(\gamma)]d\gamma\\ &+\int_{\Gamma}(\gamma\pi(\gamma)+b(\boldsymbol{\pi}(\gamma)))d\Lambda(\gamma)-\underline{U}(\Lambda(\overline{\gamma})-\Lambda(\underline{\gamma}))\end{aligned}$$

A proposed multiplier: Let us propose some nondecreasing multipliers Λ_1 and Λ_2 so that their difference, Λ, satisfies $\Lambda(\underline{\gamma})=\Lambda(\overline{\gamma})=0$ and the following conditions for all $\gamma\in(\underline{\gamma},\overline{\gamma})$:

$$\Lambda(\gamma)=\begin{cases}-w_{\pi}(\gamma,\pi_f(\gamma))f(\gamma) & p_i=0,\gamma\in\Gamma_i\\ -\lambda_i+\kappa(F(a_i)-F(\gamma)) & p_i=1,\gamma\in[a_i,\gamma_i)\cap\Gamma_i\\ -\lambda_{i+1}+\kappa(F(a_{i+1})-F(\gamma)) & p_i=1,\gamma\in[\gamma_i,a_{i+1})\cap\Gamma_i\end{cases}$$

Let us show that Condition 3 guarantees that $\kappa F(\gamma) + \Lambda(\gamma) \equiv R(\gamma)$ is non-decreasing, which then implies that $\Lambda(\gamma)$ can be written as the difference of two non decreasing functions, $R(\gamma) - \kappa F(\gamma)$. First, we show that $\kappa F(\gamma) + \Lambda(\gamma)$ is non-decreasing in the interior of each interval Γ_i. Take $i \in \{1, \ldots, n\}$. If $p_i = 0$, then this follows immediately from (c1) which states that $\kappa F(\gamma) - w_\pi(\gamma, \pi_f(\gamma))f(\gamma)$ is non decreasing for all $\gamma \in \Gamma_i$. If $p_i = 1$, we consider two cases. If $a_i < \gamma_i < a_{i+1}$, then $\kappa F(\gamma) + \Lambda(\gamma)$ is piecewise constant with one jump at γ_i. (c4) immediately implies that the jump at γ_i is non-negative. Otherwise, $\kappa F(\gamma) + \Lambda(\gamma)$ is constant for all $\gamma \in \Gamma_i$.

Next, let us show that the jumps at $\{a_i\}_{i=1}^{n+1}$ are non-negative. Take $i = 1$. We consider the following two cases. First, let $p_1 = 0$. Then it immediately follows from (c5) that the jumps at $a_1 = \underline{\gamma}$, $-w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma}) - \Lambda(\underline{\gamma})$, is non-negative. Next, consider the case where $p_1 = 1$. If $a_1 < \gamma_1$, then note that $\lambda_1 = 0$ implies that $\Lambda(\gamma)$ is continuous at $a_1 = \underline{\gamma}$ and therefore there is no jump (hence, it is non-negative). If $a_1 \geq \gamma_1$, then it follows from (c3) that $\kappa F(a_2) - \lambda_2 \geq \kappa F(a_1) + \tilde{\Lambda}(a_1)$. From the definition of $\tilde{\Lambda}(\gamma)$, we have $\tilde{\Lambda}(a_1) = 0$ if $\gamma_1 \neq a_i$ and $\tilde{\Lambda}(a_1) = -w_\pi(\underline{\gamma}, \pi_f(\underline{\gamma}))f(\underline{\gamma}) \geq 0$ if $a_1 = \gamma_1$, where the inequality follows from (c5) . Therefore, $-\lambda_2 + \kappa(F(a_2) - F(a_1)) \geq \tilde{\Lambda}(a_1) \geq 0$, which shows that the jump at a_1 is non-negative. A similar argument using (c2), (c6) and $\lambda_{n+1} = 0$ shows that the jump at $a_{n+1} = \overline{\gamma}$ is also non-negative.

Finally, let $1 < i < n$. We show that the jumps at a_i and a_{i+1} are non-negative. If $p_i = 0$, then the jumps are:

$$-\lambda_{i-1} + \kappa(F(a_{i-1}) - F(a_i)) \leq -w_\pi(a_i, \pi_f(a_i))f(a_i) \tag{38}$$

$$-\lambda_{i+2} + \kappa(F(a_{i+2}) - F(a_{i+1})) \geq -w_\pi(a_{i+1}, \pi_f(a_{i+1}))f(a_{i+1}) \tag{39}$$

Recall that $p_{i-1} = 1$ and $\gamma_{i-1} = a_i$. (c2) then implies that $\kappa F(a_{i-1}) - \lambda_{i-1} \leq \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$ for all $\gamma \in [a_{i-1}, a_i)$. Rearranging, we get $-\lambda_{i-1} + \kappa(F(a_{i-1}) - F(\gamma)) \leq \tilde{\Lambda}(\gamma)$ for all $\gamma \in [a_{i-1}, a_i)$. Since both functions are continuous for all $\gamma \in [a_{i-1}, a_i)$, this inequality holds for the limit when $\gamma \to a_i$. From the definition of $\tilde{\Lambda}(\gamma)$, we have that $\tilde{\Lambda}(\gamma) \to -w_\pi(a_i, \pi_{i-1})f(a_i)$ when $\gamma \to a_i$. Using the fact that $\pi_{i-1} = \pi_f(a_i)$, this implies that $-\lambda_{i-1} + \kappa(F(a_{i-1}) - F(a_i)) \leq -w_\pi(a_i, \pi_f(a_i))f(a_i)$, which shows that the first jump is non-negative. A similar argument using (c3) shows that the the second jump is non-negative. Next, assume $p_i = 1$. If $p_{i+1} = 0$, then by our previous argument, the jump at a_{i+1} is non-negative. If $p_{i+1} = 1$, then note that $\Lambda(\gamma)$ is continuous at a_{i+1} and therefore there is no jump (hence, it is non-negative). The same arguments hold for the jump at a_i.

Concavity of the Lagrangian: Following Amador and Bagwell (2013), we can write the Lagrangian as:

$$\mathcal{L}(\boldsymbol{\pi} \mid \Lambda) = \int_{\Gamma} [w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa(\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma)))] f(\gamma) d\gamma \\ - \int_{\Gamma} (1 - \Lambda(\gamma))\boldsymbol{\pi}(\gamma) d\gamma + \int_{\Gamma} (\gamma\boldsymbol{\pi}(\gamma) + b(\boldsymbol{\pi}(\gamma))) d(\kappa F(\gamma) + \Lambda(\gamma))$$

By the definition of κ in equation (4), we see that $w(\gamma, \boldsymbol{\pi}(\gamma)) - \kappa b(\boldsymbol{\pi}(\gamma))$ is concave in $\boldsymbol{\pi}(\gamma)$; further, we just showed that Condition 3 implies that $\kappa F(\gamma) + \Lambda(\gamma)$ is nondecreasing. Hence, the above Lagrangian is concave at the proposed multiplier.

Maximizing the Lagrangian: We now proceed to show that the proposed allocation $\boldsymbol{\pi}^*$ maximizes the Lagrangian.

For our problem, taking the Gateaux differential in direction $\boldsymbol{x} \in \hat{\Phi}$ and using that $b'(\pi_f(\gamma)) = -\gamma$ and $b'(\pi_i) = -\gamma_i$ we get that:

$$\partial\mathcal{L}(\boldsymbol{\pi}^*|\Lambda) = \\ \int_{\Gamma} [w_{\pi}(\gamma, \boldsymbol{\pi}^*(\gamma)) f(\gamma) + \Lambda(\gamma)] \boldsymbol{x}(\gamma) d\gamma + \sum_{\{i|p_i=1\}} \int_{a_i}^{a_{i+1}} [\gamma - \gamma_i] \boldsymbol{x}(\gamma) d\Lambda(\gamma)$$

which can be rewritten as:

$$\partial\mathcal{L}(\boldsymbol{\pi}^*|\Lambda) = \\ \sum_{\{i|p_i=1\}} \int_{a_i}^{a_{i+1}} [w_{\pi}(\gamma, \pi_i) f(\gamma) + \Lambda(\gamma)] \boldsymbol{x}(\gamma) d\gamma + \int_{a_i}^{a_{i+1}} [\gamma - \gamma_i] \boldsymbol{x}(\gamma) d\Lambda(\gamma) \tag{40}$$

Let us work with each interval separately. Integrating by parts over Γ_i yields:

$$\left[\int_{a_i}^{a_{i+1}} [w_{\pi}(\gamma, \pi_i) f(\gamma) + \Lambda(\gamma)]\, d\gamma\right] \boldsymbol{x}(a_{i+1}) - \int_{a_i}^{a_{i+1}} \left[\int_{a_i}^{\gamma} [w_{\pi}(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) + \Lambda(\tilde{\gamma})]\, d\tilde{\gamma}\right] d\boldsymbol{x}(\gamma) + \\ \left[\int_{a_i}^{a_{i+1}} [\gamma - \gamma_i] d\Lambda(\gamma)\right] \boldsymbol{x}(a_{i+1}) - \int_{a_i}^{a_{i+1}} \left[\int_{a_i}^{\gamma} [\tilde{\gamma} - \gamma_i] d\Lambda(\tilde{\gamma})\right] d\boldsymbol{x}(\gamma)$$

We require that the sum over the terms above be nonpositive for all nondecreasing $\boldsymbol{x}$ and zero when evaluated at $\boldsymbol{x} = \boldsymbol{\pi}^*$. Note that for $\gamma \in \Gamma_i$ such that $p_i = 1$, if $\boldsymbol{x} = \boldsymbol{\pi}^*$, then $d\boldsymbol{x}(\gamma) = 0$. So we need that:

$$\int_{a_i}^{\gamma} [w_{\pi}(\tilde{\gamma}, \pi_i) f(\tilde{\gamma}) + \Lambda(\tilde{\gamma})]\, d\tilde{\gamma} + \int_{a_i}^{\gamma} [\tilde{\gamma} - \gamma_i] d\Lambda(\tilde{\gamma}) \geq 0 \tag{41}$$

$$\forall\, \Gamma_i \text{ s.t. } p_i = 1, \forall \gamma \in \Gamma_i, \text{with equality for } \gamma = a_{i+1}.$$

We can simplify this optimality condition by integrating by parts the second term. We can then state it in the following way,

$$\int_{a_i}^{\gamma} w_\pi\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} \geq -\Lambda(\gamma)(\gamma - \gamma_i) + \Lambda(a_i)(a_i - \gamma_i) \tag{42}$$

$$\forall\, \Gamma_i \text{ s.t. } p_i = 1, \forall \gamma \in \Gamma_i, \text{with equality for } \gamma = a_{i+1}.$$

Note that the equality constraints correspond to the equality conditions in (c2) and (c3) (which are equivalent to the necessary condition (i)). To see this, note that $\Lambda(a_i) = -\lambda_i$ if $a_i < \gamma_i$ and $\Lambda(a_{i+1}) = -\lambda_{i+1}$ if $a_{i+1} > \gamma_i$.

Moreover, consider (42) for the case where $\gamma = \gamma_i$. If $\gamma_i = a_{i+1}$, then the inequality follows from the equality condition in (c2). If $a_i = \gamma_i$, the inequality trivially holds. Consider then the case where $a_i < \gamma_i < a_{i+1}$. We will show that (c2) and (c3) implies it. Assume that, contrary to (42), we have:

$$\int_{a_i}^{\gamma_i} w_\pi\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} + \lambda_i(a_i - \gamma_i) < 0.$$

Using the equality condition in (c3), we can rewrite the above inqueality as:

$$\int_{\gamma_i}^{a_{i+1}} w_\pi\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} - \lambda_{i+1}(a_{i+1} - \gamma_i) > 0.$$

It then follows that $\tilde{\Lambda}(\gamma) \longrightarrow -\infty$ as $\gamma \longrightarrow \gamma_i^-$. This clearly contradicts (c2) since, then, one can find γ sufficiently close to γ_i such that $\tilde{\Lambda}(\gamma) + \kappa F(\gamma) < \kappa F(a_i) - \lambda_i$.

Next, consider (42) for the case where $\gamma \neq \gamma_i$.

If $\gamma > \gamma_i$, we can isolate $\Lambda(\gamma)$ in (42), which yields:

$$\Lambda(\gamma) \geq \tilde{\Lambda}(\gamma), \forall\, \Gamma_i \text{ s.t. } p_i = 1, \forall \gamma > \gamma_i$$

where $\Lambda(\gamma) = -\lambda_{i+1} + \kappa(F(a_{i+1}) - F(\gamma))$ for all $\gamma \in \Gamma_i \cap (\gamma_i, a_{i+1}]$. It is clear that the above inequality is implied by (c3) which states that for $\gamma \in \Gamma_i \cap (\gamma_i, a_{i+1}]$, $\kappa F(a_{i+1}) - \lambda_{i+1} \geq \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$.

If $\gamma < \gamma_i$, then we can rewrite (42) using the equality condition in (c2) in the following way:

$$-\int_{\gamma}^{a_{i+1}} w_\pi\left(\tilde{\gamma}, \pi_i\right) f(\tilde{\gamma}) d\tilde{\gamma} + \lambda_{i+1}(a_{i+1} - \gamma_i) \geq -\Lambda(\gamma)(\gamma - \gamma_i) \tag{43}$$

We can then isolate $\Lambda(\gamma)$ in (43), which yields:

$$\Lambda(\gamma) \leq \tilde{\Lambda}(\gamma), \forall \Gamma_i \text{ s.t. } p_i = 1, \forall \gamma < \gamma_i$$

where $\Lambda(\gamma) = -\lambda_i + \kappa(F(a_i) - F(\gamma))$ for all $\gamma \in \Gamma_i \cap [a_i, \gamma_i)$. Once again, it is clear that the above inequality is implied by (c2) which states that for $\gamma \in \Gamma_i \cap [a_i, \gamma_i)$, $\kappa F(a_i) - \lambda_i \leq \kappa F(\gamma) + \tilde{\Lambda}(\gamma)$. $\square$